

MATH606: Complex Algebraic Geometry

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1. Introduction: From Complex Analysis to Algebraic Geometry

“Complex analysis is more rigid, even magical, when compared to real analysis.”

For example, if $U \subset \mathbb{C}$ is open, and $f : U \rightarrow \mathbb{C}$ has a single complex derivative at each point $x \in U$, then f is smooth (has infinitely many derivatives) and analytic (locally representable by a power series). Since power series are close to polynomials, it's not surprising that the local study of power series (analysis) and polynomials (geometry) is similar.

Definition

1.1

A **complex analytic set** is defined locally as

$$\{x \in U \mid f_1(x) = \cdots = f_m(x) = 0\},$$

where $f_i : U \rightarrow \mathbb{C}$ are holomorphic.

A **complex algebraic set** is defined in the same way, replacing “holomorphic” by “polynomial”.

1.1. Classification of Real and Complex Manifolds

Real manifolds were defined and studied as sub-manifolds of $\mathbb{R}^n$ long before their definitions using charts and atlases. We can think of “embedding theorems” as a way of seeing an abstract manifold X by embedding it in coordinate space, writing down global equations for it.

Whitney's Theorem, ~1935

1.2

If M is a smooth manifold of dimension n , then there exists a smooth (C^∞) embedding $f : M \hookrightarrow \mathbb{R}^{2n+1}$.

Suppose X is a complex manifold (more generally, a complex analytic space, which is a space locally defined by a system of holomorphic functions). Then, there exists a holomorphic embedding $f : X \hookrightarrow \mathbb{C}^N$ for some $N \iff X$ is a Stein manifold.

Definition

1.3

A complex manifold X is **Stein** if for any discrete subset $\{x_n\}_{n \geq 1}$ for points in X , and sequence $\{c_n\}_{n \geq 1}$ of complex numbers, there exists a holomorphic function $f : X \rightarrow \mathbb{C}$ such that $f(x_n) = c_n$ for all $n \geq 1$.

Theorem

1.4

The only compact complex manifolds that embed holomorphically in $\mathbb{C}^N$ are points.

Proof. We have the projection maps $z_i : X \rightarrow \mathbb{C}$ for $i = 1, \dots, N$. Because X is compact, z_i is a closed map. By the open mapping theorem, if z_i is not constant, then z_i is open. This is impossible since $\mathbb{C}$ is connected, so $z_i = c_i$ is constant for all i . Hence, $X = \{(c_1, \dots, c_N)\}$ is a point. $\square$

Since $\mathbb{C}^N$ can't serve as an embedding space for compact complex manifolds, we can try to use

projective space $\mathbb{P}^N$ instead. On $\mathbb{P}^N = \{[z_0 : \cdots : z_n]\}$ we have homogeneous coordinates z_i , and analytic/algebraic subsets can be defined using homogeneous polynomials.

Chow's Theorem, 1949

1.5

If a complex manifold (or analytic space) embeds holomorphically as a complex submanifold of $\mathbb{P}^N$ for some N , then X is a smooth algebraic variety. That is,

$$X = \{[z] \in \mathbb{P}^N \mid f_1(z) = \cdots = f_m(z) = 0\},$$

where $f_1, \dots, f_m$ are polynomials in $z_0, \dots, z_n$

Remark 1.6. *There is a 1-variable analogue of this result: if $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function such that $\lim_{z \rightarrow \infty} f(z)$ exists in $\mathbb{C} \cup \{\infty\}$, then f is a polynomial.*

The condition on f means that $f : \mathbb{C} \rightarrow \mathbb{C}$ extends to a holomorphic function $\tilde{f} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, where $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$

Soon after, Serre proved comparison theorems for sheaves and sheaf cohomology on projective varieties.

Serre's "GAGA" Theorem, 1955, Géométrie Algébrique et Géométrie Analytique

1.7

If $\mathcal{F}$ is a coherent analytic sheaf on a projective variety X_{hol} , there is a unique coherent algebraic sheaf $\mathcal{F}_{\text{alg}}$ on X_{alg} such that $\mathcal{F} = (\mathcal{F}_{\text{alg}})_{\text{hol}}$. Moreover, the natural maps

$$H^i(X_{\text{alg}}, \mathcal{F}_{\text{alg}}) \rightarrow H^i(X_{\text{hol}}, \mathcal{F})$$

are isomorphisms for all $i \geq 0$.

Definition

1.8

A **Riemann surface** is a 1-dimensional complex manifold X . Underlying X is a 2-dimensional orientable real manifold $X_{\mathbb{R}}$. If X is compact, then $X_{\mathbb{R}}$ is diffeomorphic to a g -torus, which we represent as

$$H_1(X_{\mathbb{R}}, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$$

Consider an irreducible polynomial $p \in \mathbb{C}[z, w]$, and the subset of $\mathbb{C}^2$ given by

$$V(p) := \{(z, w) \in \mathbb{C}^2 \mid p(z, w) = 0\}.$$

In general, $V(p) \subset \mathbb{C}^2$ is neither compact nor a smooth manifold. Riemann showed that $V(p)$ can be modified to obtain a compact complex manifold $X(p)$, which is the Riemann surface of the algebraic function $p(z, w)$. Moreover, $X(p)$ comes with two meromorphic functions f, g which extend z and w .

Later, Riemann defined abstract manifolds. We obtain compact Riemann surfaces X , which do not obviously come from polynomials p , but it turns out that these are all algebraic varieties. This was shown by studying the *function theory* on X . For every compact X ,

$$\mathcal{O}(X) := \{f : X \rightarrow \mathbb{C} \text{ holomorphic}\} \simeq \mathbb{C}$$

is just the constant functions. Instead, we consider the field $m(X)$ of meromorphic functions on X (these are just holomorphic functions $f : X \rightarrow \mathbb{P}^1$).

Example 1.9. For the projective line, $m(\mathbb{P}^1) \simeq \mathbb{C}(z)$, where z is the affine coordinate on $\mathbb{C} \subset \mathbb{P}^1$.

Studying meromorphic functions instead of holomorphic functions makes things more interesting. For instance, we'll prove the following theorem later, which is rather difficult.

Theorem

1.10

If X is a compact Riemann surface, then there exists a non-constant meromorphic $\pi : X \rightarrow \mathbb{P}^1$.

Consider the sequence

$$X \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{z} \mathbb{C}.$$

Pulling back z via π to $m(X)$ gives an inclusion of fields

$$\mathbb{C}(z) \xrightarrow{\sim} \mathbb{C}(\pi) \hookrightarrow \mathcal{M}(X).$$

So, $\mathcal{M}(X) \supset \mathbb{C}(z)$ is a field extension, which turns out to be finite. Thus, the primitive element theorem yields

$$\mathcal{M}(x) = \mathbb{C}(\pi)(g).$$

Let $\tilde{p} \in \mathbb{C}(\pi)[z]$ be the minimal polynomial of g . Then, clearing denominators, we get a polynomial p with $p(\pi, g) = 0$. Moreover, X is the Riemann surface associated to p . We therefore see that X is determined by $m(X)$, an “algebraic function field in one variable”. This is the point of view in algebraic geometry, where X is a smooth algebraic curve over $\mathbb{C}$.

1.2. The Riemann-Roch Problem

Given a compact Riemann surface X , let $f \in m(X)$ be meromorphic, and

$$D := \operatorname{div}(f) = \text{“Zeros}(f)\text{”} - \text{“Poles}(f)\text{”}.$$

More formally,

$$D \in \bigoplus_{p \in X} \mathbb{Z}[p].$$

Note that we must have $\deg D = 0$, where

$$\deg D = \deg \left(\bigoplus n_p [p] \right) = \sum n_p.$$

In fact, D determines f up to scalar. Indeed, if $\operatorname{div} f = \operatorname{div} g$, then $\frac{f}{g} \in \mathcal{O}(X)$ is non-zero and holomorphic on X , and is therefore constant.

Riemann's Idea. Given any divisor D , consider the complex vector space

$$H^0(D) := \{f \in m(X) \mid \operatorname{div} f \geq -D\}.$$

Then, $\dim_{\mathbb{C}} H^0(D) < \infty$.

Riemann-Roch Problem. Compute $\dim_{\mathbb{C}} H^0(D)$.

Riemann showed

$$\dim H^0(D) \geq 1 - g + \deg D.$$

This problem in its entirety is still open, and is a motivating problem for this course.

2. Riemann Surfaces

Definition

2.1

Let X be a 2-dimensional real topological manifold. We consider pairs (U, φ) where $U \subset X$ is open, and $\varphi : U \rightarrow \mathbb{C}$ is a homeomorphism on its image. Two such pairs (U_1, φ_1) and (U_2, φ_2) are said to be **compatible** if

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$$

is holomorphic. In this case, its inverse is also holomorphic.

A **complex structure** on X is a family of pairs $\{(U_i, \varphi_i)\}_{i \in I}$ which are compatible, and $X = \bigcup_i U_i$. We typically assume such a family is maximal. The pairs (U_i, φ_i) are called **holomorphic charts**. In a coordinate neighborhood U , we usually identify U with $\varphi(U)$, and write z for the map φ as one does with 1 complex variable in $\mathbb{C}$.

A **Riemann surface** is a connected 2-dimensional real manifold with a complex structure on it. Given a Riemann surface X , a function $f : X \rightarrow \mathbb{C}$ is **holomorphic** if f is continuous, and for every chart (U, φ) of X ,

$$f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{C}$$

is holomorphic. This also defines holomorphic functions on open subsets $\Omega \subset X$. Let $\mathcal{O}(\Omega)$ denote the $\mathbb{C}$ -algebra of all holomorphic functions on Ω . If X, Y are Riemann surfaces, and $f : X \rightarrow Y$ is continuous, we say that f is **holomorphic** if for every chart (V, ψ) on Y , the function

$$\psi \circ f : f^{-1}(V) \rightarrow \psi(V)$$

is holomorphic. A holomorphic map $f : X \rightarrow Y$ is an **analytic isomorphism** (or **biholomorphism**) if there exists a holomorphic map $g : Y \rightarrow X$ such that $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$.

Remark 2.2. A theorem of Rado asserts that any Riemann surface X is a 2nd-countable topological space, so has a countable basis for its topology.

Example 2.3. Any connected open set Ω in $\mathbb{C}$ is a Riemann surface. Similarly, any connected open subset Ω of a Riemann surface X is a Riemann surface.

Example 2.4 (The Riemann sphere $\mathbb{P}^1$). The **Riemann sphere** $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is the one-point compactification of $\mathbb{C}$, where the open neighborhoods of ∞ are complements of compact subsets $K \subset \mathbb{C}$. With this topology, $\mathbb{P}^1$ is homeomorphic to the 2-sphere $S^2 \subset \mathbb{R}^3$. Set $U_1 := \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$ and

$U_2 := \mathbb{P}_1 \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}$, and define the charts $\varphi_i : U_i \rightarrow \mathbb{C}$ by $\varphi_1 = \text{id}_{\mathbb{C}}$, and

$$\varphi_2(z) = \begin{cases} 1/z & z \in \mathbb{C}^*, \\ 0 & z = \infty. \end{cases}$$

Then,

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) = \mathbb{C}^* \rightarrow \mathbb{C}^* = \varphi_2(U_1 \cap U_2)$$

sends $z \mapsto 1/z$, and is a biholomorphic map on $\mathbb{C}^*$.

Example 2.5 (Complex tori). Let ω_1 and ω_2 be non-zero complex numbers, which are linearly independent over $\mathbb{R}$. Define $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ to be a **lattice** in $\mathbb{C}$, and let $\pi : \mathbb{C} \twoheadrightarrow \mathbb{C}/\Lambda$ be the canonical projection, giving $\mathbb{C}/\Lambda$ the quotient topology. Since π is a covering map, $\mathbb{C}/\Lambda$ inherits a complex structure from the complex structure on $\mathbb{C}$. Note that $\mathbb{C}/\Lambda$ is compact, and is homeomorphic to the real 2-torus $S^1 \times S^1$.

More explicitly, let $\mathbb{C} = \bigcup V_i$ be an open cover of $\mathbb{C}$ by disks such that $\pi|_{V_i}$ is a homeomorphism onto $U_i := \pi(V_i) \subset \mathbb{C}/\Lambda$. Let $\varphi_i : U_i \rightarrow V_i \subset \mathbb{C}$ be the inverse $(\pi|_{V_i})^{-1}$. Then, the family $\{(U_i, \varphi_i)\}$ is a complex structure on $X := \mathbb{C}/\Lambda$. Indeed, let $x_0 \in U_i \cap U_j$, $z_i := \varphi_i(x_0)$, and $z_j := \varphi_j(x_0)$. Then, $\lambda := z_i - z_j \in \Lambda$, and

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j)$$

sending $z \mapsto z + \lambda$ is biholomorphic.

Riemann Extension Theorem

2.6

Let X be a Riemann surface, and $a \in X$. Let $f \in \mathcal{O}(X \setminus \{a\})$ be such that f is bounded on $U \setminus \{a\}$ for some neighborhood U of a . Then, there exists a function $F \in \mathcal{O}(X)$ such that $F|_{X \setminus \{a\}} = f$.

Principle of Analytic Continuation

2.7

Let X and Y be Riemann surfaces, and $f : X \rightarrow Y$ be holomorphic. If there is a non-empty open set $U_0 \subset X$ such that $f|_{U_0}$ is a constant $y_0 \in Y$, then $f(x) = y_0$ for all $x \in X$.

Proof. Let

$$E := \{x \in X \mid \exists U \ni x \text{ open, } f(U) = \{y_0\}\}.$$

Then, E is open and $U_0 \subset E$. Let $x_0 \in \bar{E}$ and $(U, \varphi), (V, \psi)$ be charts on X and Y respectively with $x_0 \in U$ and $f(x_0) \in V$. Let Ω be the connected component of $\varphi(U \cap f^{-1}(V))$ containing $\varphi(x_0)$. Now, $\varphi(E \cap U) \cap \Omega$ is a non-empty open set. Moreover, $F := \psi \circ f \circ \varphi^{-1}|_{\Omega}$ is holomorphic on Ω , and

$$F(\varphi(E \cap U) \cap \Omega) = \{\psi(y_0)\}.$$

Thus, by analytic continuation on $\mathbb{C}$, $F(\Omega) = \{\psi(y_0)\}$, hence $f(x) = y_0$ for all x in a neighborhood of x_0 . Therefore, $x_0 \in E$, so E is closed. Since X is connected, and E is both open and closed, we must have $E = X$. $\square$

Exercise 1. Show that the principle of analytic continuation stated above is equivalent to the formulation involving accumulation points in X .

Definition

2.8

A **meromorphic function** on a Riemann surface X is a holomorphic function $f : X' \rightarrow \mathbb{C}$, where $X' \subset X$ is an open subset such that

- (i) $X \setminus X'$ is discrete in X ,
- (ii) For any point $p \in X'$,

$$\lim_{x \rightarrow p} |f(x)| = \infty$$

The points of $X \setminus X'$ are called the **poles** of f . Let $\mathcal{M}(X)$ denote the $\mathbb{C}$ -algebra of all meromorphic functions on X . We'll soon show that $\mathcal{M}(X)$ is a field.

Remark 2.9. If (U, z) is a coordinate neighborhood of a pole p of f , with $z(p) = 0$, then f has a Laurent expansion

$$f(z) = \sum_{v=-k}^{+\infty} c_v z^v$$

in a neighborhood of p , with $c_{-k} \neq 0$. The integer $-k := \text{ord}_p(f)$ is well-defined, and independent of the choice of local coordinates around p .

Our definition of meromorphic functions does not work in higher dimensions, since isolated zeros/poles don't exist in complex dimension $n \geq 2$. The following is a better, more generalized definition:

Definition

2.10

A **meromorphic function** f is one that locally can be written as $\frac{g}{h}$, where g and h are holomorphic functions.

Theorem

2.11

Let X be a Riemann surface, and $f \in \mathcal{M}(X)$. For each pole p of f , let $f(p) := \infty$. Then, the extended function

$$f : X \rightarrow \mathbb{P}^1$$

is holomorphic. Conversely, if $f : X \rightarrow \mathbb{P}^1$ is holomorphic, either f is identically ∞ , or $f^{-1}(\infty)$ consists of isolated points in X , and

$$f : X \setminus f^{-1}(\infty) \rightarrow \mathbb{C}$$

is a meromorphic function on X . Hence, we can identify $f \in \mathcal{M}(X)$ with the corresponding holomorphic map $f : X \rightarrow \mathbb{P}^1$

Proof.

($\implies$) Let $f \in \mathcal{M}(X)$ be meromorphic, and let P be the set of poles of f . Then, the induced map $f : X \rightarrow \mathbb{P}^1$ is continuous. Suppose $\varphi : U \rightarrow V$ and $\psi : U' \rightarrow V'$ are holomorphic charts on X and

$\mathbb{P}^1$ with $f(U) \subset U'$. We must show that $g := \psi \circ f \circ \varphi^{-1} : V \rightarrow V'$ is holomorphic. Indeed, since $f \in \mathcal{O}(X \setminus P)$, we get $g \in \mathcal{O}(V \setminus \varphi(P))$. By the Riemann Extension Theorem ([Theorem 2.6](#)), g is holomorphic on all of V .

($\Leftarrow$) Easy consequence of analytic continuation. $\square$

Corollary

2.12

Analytic continuation also holds for meromorphic functions on a Riemann surface X . So, any $f \in \mathcal{M}(X)$ which is not identically zero has isolated zeros on X . This implies that $1/f \in \mathcal{M}(X)$, thus $\mathcal{M}(X)$ is a field.

Local Behavior of Holomorphic Maps

2.13

Suppose X, Y are Riemann surfaces, and $f : X \rightarrow Y$ is a non-constant holomorphic map. Let $a \in X$ and $b = f(a)$. Then, there exists $k \in \mathbb{Z}^+$ and charts $\varphi : U \rightarrow V$ on X , $\psi : U' \rightarrow V'$ on Y , such that

- (i) $a \in U, \varphi(a) = 0, b \in U', \psi(b) = 0, f(U) \subset U'$.
- (ii) $F := \psi \circ f \circ \varphi^{-1} : V \rightarrow V'$ is $F(z) = z^k$ for all $z \in V$.

Proof. Charts (U, φ) and (U', ψ) satisfying (i) clearly exist, and the induced map F of (ii) is non-constant by analytic continuation. Hence, the question is local, and it suffices to show that if $f : \Omega \rightarrow \mathbb{C}$ is holomorphic and non-constant in a neighborhood Ω of $0 \in \mathbb{C}$, then there exists a neighborhood U of 0 and $\psi : U \rightarrow U'$ biholomorphic such that $f \circ \psi(z) = z^k$ for some $k \geq 1$.

Since f is holomorphic and non-constant, we may write $f(z) = z^k g(z)$ for some $k \geq 1$, and $g \in \mathcal{O}(0)$ with $g(0) \neq 0$. We claim that there exists a neighborhood U'' of 0 such that $g(z) = h(z)^k$ for some $h \in \mathcal{O}(U'')$, with $h(0) \neq 0$. Indeed there exists $\tilde{g}$ such that

$$\begin{array}{ccc} & & \mathbb{C} \\ & \nearrow \tilde{g} & \downarrow \exp \\ D & \xrightarrow{g} & \mathbb{C}^* \end{array}$$

commutes. Let $h := \exp(\tilde{g}/k)$. Then, $\varphi(z) = zh(z)$ is a biholomorphic map for some neighborhood U' of 0 , $U' \subset U''$, onto its image U such that

$$f \circ \varphi^{-1}(z) = z^k$$

for all $z \in U$ $\square$

Remark 2.14. The integer k may be characterized as follows: for any neighborhood U_0 of a , there exists neighborhoods $U \subset U_0$ of a and W of $b = f(a)$ such that $\#f^{-1}(y) \cap U = k$ for all $y \in W, y \neq b$. This k is the **multiplicity** of f at a .

Notation. If $K \subset \mathbb{C}$ is compact, we write $\mathcal{O}(K)$ to mean functions $f : K \rightarrow \mathbb{C}$ which are holomorphic in an open neighborhood of K .

Corollary**2.15**

Let $f : X \rightarrow Y$ be a non-constant holomorphic map. Then, f is an open map.

Proof. By [Theorem 2.13](#), it is clear that if U is a neighborhood of $a \in X$, then $f(U)$ is a neighborhood of $f(a)$. $\square$

Theorem**2.16**

Suppose that X and Y are Riemann surfaces with X compact and Y connected, and $f : X \rightarrow Y$ is a non-constant holomorphic map. Then, f is surjective, and Y is compact.

Proof. By [Corollary 2.15](#), f is an open map, hence $f(X)$ is both an open and closed subset of Y which is connected. Hence, $f(X) = Y$, and Y is compact. $\square$

Corollary**2.17**

If X is a compact Riemann surface, and $f : X \rightarrow \mathbb{C}$ is holomorphic, then X is constant.

Corollary (Maximum Principle)**2.18**

Let X be a Riemann surface, and $f : X \rightarrow \mathbb{C}$ a non-constant holomorphic map. Then, $|f|$ does not attain a maximum.

Proof. As f is non-constant, $f(X)$ is an open subset of $\mathbb{C}$, so if $f(X) \subset \overline{D}(x_0, R)$ for some $x_0 \in \mathbb{C}, R > 0$, then $f(X) \subset D(x_0, R)$. So, the complex maximum principle applies. $\square$

Corollary**2.19**

Every meromorphic function on $\mathbb{P}^1$ is rational, that is, can be written as a quotient of two polynomials.

Proof. If $f \in \mathcal{M}(\mathbb{P}^1)$, then f induces a holomorphic map $\tilde{f} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ by [Theorem 2.11](#), so f has finitely many poles. We may assume ∞ is not a pole of f , otherwise consider $1/f$ instead of f .

Let $a_1, \dots, a_n \in \mathbb{C}$ be the poles of f , and

$$h_\nu(z) := \sum_{j=-k_\nu}^{-1} c_{\nu_j}(z - a_\nu)^j$$

be the principle part of f at the pole a_ν , for $\nu = 1, 2, \dots, n$. Then,

$$f - h_1 - h_2 - \dots - h_n$$

is holomorphic on $\mathbb{P}^1$, and is therefore a constant. $\square$

We've shown that $\mathcal{M}(\mathbb{P}^1) \simeq \mathbb{C}(z)$, the field of rational functions in z .

Exercise 2. Deduce that

$$\text{Aut}(\mathbb{P}^1) := \{f : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \text{ biholomorphic}\} \simeq \text{PGL}_2(\mathbb{C}) := \text{GL}_2(\mathbb{C}) / Z(\text{GL}_2(\mathbb{C}))$$

3. Covering Maps

In this section, X and Y are topological manifolds.

Definition

3.1

Let X and Y be topological manifolds (more generally, locally compact Hausdorff spaces). A continuous map $f : X \rightarrow Y$ is **proper** if $f^{-1}(K)$ is compact for any compact $K \subset Y$. We say that f is a **local homeomorphism** if for all $x \in X$, there exists an open neighborhood $U \ni x$ such that $f(U)$ is open in Y , and $f|_U : U \rightarrow f(U)$ is a homeomorphism. We say that f is a **covering map** if for all $y \in Y$, there exists a neighborhood $V \ni y$ which is **evenly covered** by f , meaning

$$f^{-1}(V) = \bigsqcup_{j \in I} U_j$$

is a disjoint union of open sets U_j such that $f|_{U_j} : U_j \rightarrow V$ is a homeomorphism onto V , for all $j \in I$. We say X is a **covering** of Y if a covering map exists.

Remark 3.2. 1. Since the cardinality $|I|$ of $f^{-1}(y)$ is a locally constant function on Y for any covering map f , if Y is connected then $\#f^{-1}(y)$ is independent of $y \in Y$. The covering is said to be **finite** (resp. **infinite**) if $\#f^{-1}(y)$ is constant (resp. infinite). The covering is said to be an **n -sheeted covering** if $\#f^{-1}(y) = n$.

2. If $f : X \rightarrow Y$ and $\tilde{f} : \tilde{X} \rightarrow Y$ are two coverings of Y , they are said to be **isomorphic coverings** if there exists a homeomorphism $\varphi : \tilde{X} \rightarrow X$ such that $f \circ \varphi = \tilde{f}$. That is, the diagram

$$\begin{array}{ccc} X & \xleftarrow{\varphi} & \tilde{X} \\ & \searrow f & \swarrow \tilde{f} \\ & Y & \end{array}$$

commutes.

3. Any covering map is a local homeomorphism, but the converse is false. For example, the inclusion $i : \mathbb{C}^* \hookrightarrow \mathbb{C}$ is clearly a local homeomorphism, and clearly not a covering map.

Example 3.3. Any n -sheeted covering of $\mathbb{D}^* := \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ is isomorphic to $f_n(z) := z^n$.

Lemma

3.4

If $f : X \rightarrow Y$ is proper, then f is closed.

Proof. If $F \subset X$ is closed, and $K \subset Y$ is compact, it suffices to show $f(F) \cap K$ is closed. Indeed, $f(F) \cap K = f(F \cap f^{-1}(K))$, and since f is proper we know $f^{-1}(K)$ is compact, hence $F \cap f^{-1}(K)$ is compact. Thus, the image of $F \cap f^{-1}(K)$ is closed in K . $\square$

Proposition

3.5

Suppose $p : X \rightarrow Y$ is a local homeomorphism with Y connected. Then, p is a finite covering map if and only if p is proper.

Proof.

($\implies$) If p is a finite covering, $y_0 \in Y$, and $V \supset y_0$ evenly covered, then $p|_{p^{-1}(V)} : p^{-1}(V) \rightarrow V$ is clearly proper, since p is a finite cover. In general, if K is any compact set, then K can be covered by finitely many compact neighborhoods, each evenly covered by p , so it follows that p is proper.

($\impliedby$) Let p be a proper local homeomorphism. Then, p is closed by [Lemma 3.4](#), so $p(X)$ is both open and closed in Y , which is connected, hence p is surjective. Now, let $y_0 \in Y$ and $p^{-1}(y_0) = \{x_1, \dots, x_n\}$. Let U'_j be a neighborhood of x_j such that $p|_{U'_j} : U'_j \rightarrow p(U'_j) =: V_j$ is a homeomorphism, and the U'_j are disjoint.

We want to find our evenly covered neighborhood. If $X \setminus \bigcup U'_j$ is closed in X , and p is proper, then $F := p(X \setminus \bigcup U'_j)$ is closed in Y , and $y_0 \notin F$. Hence, $V' := Y \setminus F$ is an open neighborhood of y_0 , and $p^{-1}(V') \subset \bigsqcup U'_j$. Define $V := V' \cap V_1 \cap \dots \cap V_n$. Then, V is our desired neighborhood: it is an open neighborhood such that $p^{-1}(V) \subset U'_1 \cup \dots \cup U'_n$. For $1 \leq i \leq n$, define

$$U_i := p^{-1}(V) \cap U'_i = \left(p|_{U'_i}\right)^{-1}(V).$$

Then, U_i is mapped homeomorphically onto V by p , and $U_1, \dots, U_n$ are disjoint. So, V is evenly covered by p . $\square$

4. Branched and Unbranched Coverings

Definition

4.1

Let $f : X \rightarrow Y$ be a non-constant holomorphic map of Riemann surfaces. A point $x \in X$ is called a **branch point** or **ramification point** of f if there is no neighborhood U of x such that $f|_U$ is injective. We say that f is an **unbranched holomorphic map** if it has no branch points. This happens if and only if f is a local homeomorphism.

Remark 4.2. The set of branch points of f in X is closed and discrete in X .

Definition

4.3

Assume $f : X \rightarrow Y$ is a proper map, and let A be the set of all branch points of f . Then, the set $B := f(A)$ is also closed and discrete. B is called the set of **critical values** of f .

Let $Y' := Y \setminus B$ and $X' := X \setminus f^{-1}(B) \subset X \setminus A$. Then, $f|_{X'} : X' \rightarrow Y'$ is a proper unbranched holomorphic covering map, with a well-defined finite number of sheets in n (that is, every $c \in Y'$ is achieved exactly n times). We will extend this statement to include the critical values $b \in B$ by considering multiplicities. For $x \in X$, let $v(f, x)$ be the multiplicity with which f takes the value $f(x)$ at the point x . We say that f takes the value $c \in Y$, counting multiplicities, m times on X , if

$$m := \sum_{x \in f^{-1}(c)} v(f, x)$$

Theorem

4.4

Suppose that X and Y are two Riemann surfaces, and $f : X \rightarrow Y$ is a proper, non-constant, holomorphic map. Then, there exists $n \in \mathbb{Z}^+$ such that f takes every value $c \in Y$, counting multiplicities, n times.

Proof. Let n be the number of sheets of the unbranched cover $f : X' \rightarrow Y'$ defined earlier. Suppose $b \in B$ is a critical value, $f^{-1}(b) = \{x_1, \dots, x_r\}$, and let $k_j := v(f, x_j)$ for $j = 1, \dots, r$. Then, there exists neighborhood U_j of x_j , and V_j of b , such that for all $c \in V_j \setminus \{b\}$, the set $f^{-1}(c) \cap U_j$ has cardinality k_j for $1 \leq j \leq r$.

Claim 4.5. We can find a neighborhood $V \subset V_1 \cap \dots \cap V_r$ of b such that $f^{-1}(V) \subset U_1 \cap \dots \cap U_r$.

Proof. Indeed, f is proper, hence closed, so $f(X \setminus (U_1 \cup \dots \cup U_r))$ is a closed set in Y not containing b . So, take $V := V' \cap V_1 \cap \dots \cap V_r$, where $V' := Y \setminus f(X \setminus (U_1 \cup \dots \cup U_r))$. $\square$

Now, for every $c \in V \cap Y'$, $f^{-1}(c)$ consists of $k_1 + \dots + k_r$ points, and this number is exactly n since $\#f^{-1}(c) = n$ for all $c \in Y'$. $\square$

Corollary

4.6

Let X be a compact Riemann surface. Then, every non-constant meromorphic function $f : X \rightarrow \mathbb{P}^1$ has as many zeros as poles, where each is counted with multiplicities.

Proof. f is proper, so [Theorem 4.4](#) applies. $\square$

We want to study $\mathcal{M}(X)$ for any compact Riemann surface X , generalizing $\mathcal{M}(\mathbb{P}^1) \simeq \mathbb{C}(z)$. The main difficulty is that if X is a compact Riemann surface, we can construct any non-constant $f \in \mathcal{M}(X)$. We examine the case of $\mathcal{M}(\mathbb{C} \setminus \Lambda)$ where $\Lambda \subset \mathbb{C}$ is a lattice. These are elliptic functions.

Definition

4.7

A meromorphic function f on $\mathbb{C}$ is said to be **elliptic** if there exists a lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ in $\mathbb{C}$ such that $f(z + \omega) = f(z)$ for all $z \in \mathbb{C} \setminus \Lambda$ and $\omega \in \Lambda$. These correspond to meromorphic functions on the torus $\mathbb{C}/\Lambda$.

5. Elliptic Functions

Definition

5.1

Let $\Omega := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. A meromorphic function $f : \mathbb{C} \rightarrow \mathbb{P}^1$ is **elliptic** with respect to Ω if $f(z + \omega) = f(z)$ for all $\omega \in \Omega$. Such a function corresponds to a holomorphic map $f : T \rightarrow \mathbb{P}^1$, where $T := \mathbb{C}/\Omega$.

Definition

5.2

The **order** $\text{ord}(f)$ of an elliptic function f is the number of solutions $f(z) = \infty$, i.e. the number of poles of f on T , counted with multiplicity.

Remark 5.3. An elliptic function f is constant $\iff \text{ord}(f) = 0$.

Assume that P is a fundamental parallelogram for Ω with vertices at $t, t + \omega_1, t + \omega_1 + \omega_2$, and $t + \omega_2$, where $t \in \mathbb{C}$ is such that ∂P has no zeros or poles of f . In the homework, we showed

$$\int_{\partial P} f(z) dz = 0,$$

so by the residue theorem, the sum of the residues of f inside P is 0.

Corollary

5.4

There are no elliptic functions of order $N = 1$.

Corollary

5.5

If an elliptic function has order $N > 0$, then f takes every value $c \in \mathbb{P}^1$ exactly N times on P , counting multiplicities.

Corollary

5.6

Let f, g be elliptic functions with respect to Ω , with zeroes and poles of the same orders, and at the same points. Then, $f(z) = \lambda g(z)$ for some $\lambda \in \mathbb{C}^*$.

Proof. The map $f/g : T \rightarrow \mathbb{P}^1$ is holomorphic on compact Riemann surfaces, hence constant. $\square$

Theorem

5.7

Let the congruence classes mod Ω of zeros and poles of an elliptic function f be $[a_1], \dots, [a_r]$ and $[b_1], \dots, [b_s]$, with multiplicities $k_1, \dots, k_r$ and $\ell_1, \dots, \ell_s$ respectively. Then,

$$\sum_{j=1}^r k_j a_j \equiv \sum_{j=1}^s \ell_j b_j \pmod{\Omega}.$$

This means we cannot just choose any set of zeros/poles, and assume an elliptic function has them.

6. The Weierstrass $\wp$ -function

Lemma

6.1

If $s \in \mathbb{R}$, then

$$\sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{|\omega|^s} \text{ converges } \iff s > 2$$

Proof. Let

$$P_r := \{a\omega_1 + b\omega_2 \mid a, b \in \mathbb{R}, \max\{|a|, |b|\} = r\}$$

for $r \in \mathbb{Z}^+$, and $\Omega_r := \Omega \cap P_r$. Then,

$$\Omega = \{0\} \sqcup \bigsqcup_{r \geq 1} \Omega_r,$$

and $|\Omega_r| = 8r$. Let m and M be the smallest and largest moduli of all elements of P_1 . Then, since $\Omega_r \subset P_r = r \cdot P_1$, we have

$$rM \geq |\omega| \geq rm, \quad \forall \omega \in \Omega_r.$$

Thus,

$$\sigma_r := \sum_{\omega \in \Omega_r} \frac{1}{|\omega|^s} \implies 8r(rM)^{-s} \leq \sigma_r \leq 8r(rm)^{-s},$$

thus

$$\sum_{r=1}^{\infty} \sigma_r < \infty \iff \sum_{r=1}^{\infty} r^{1-s} < \infty \iff s > 2.$$

□

Corollary

6.2

For each integer $N \geq 3$, the function

$$F_N(z) := \sum_{\omega \in \Omega} \frac{1}{(z - \omega)^N}$$

is elliptic of order N .

Proof. Let K be a compact subset of $\mathbb{C} \setminus \Omega$. Then, for all but finitely many $\omega \in \Omega$, say for all $\Omega' \subset \Omega$, we have $|\omega| \geq 2z$ for all $z \in K$. Hence,

$$|z - \omega| \geq |\omega| - |z| \geq \frac{1}{2}|\omega|, \quad \forall z \in K, \forall \omega \in \Omega'.$$

Thus,

$$\left\| \frac{1}{(z - \omega)^N} \right\|_K \leq 2^N \frac{1}{|\omega|^N}, \quad \forall \omega \in \Omega'.$$

Hence, if $N \geq 3$, then by the previous lemma,

$$\sum_{\omega \in \Omega} \left\| \frac{1}{(z - \omega)^N} \right\|_K \text{ converges,}$$

so $F_N(z)$ converges to a meromorphic function with a pole of order N at each $\omega \in \Omega$, while $F_N|_{\mathbb{C} \setminus \Omega}$ is holomorphic. Since the series converges absolutely, it can be rearranged, so if $\omega_0 \in \Omega$,

$$F_n(z + \omega_0) = \sum_{\omega \in \Omega} \frac{1}{(z + \omega_0 - \omega)^N} = \sum_{\omega' \in \Omega} \frac{1}{(z - \omega')^N} = F_N(z)$$

and we're done. □

Definition

6.3

The **Weierstrass $\wp$ -function** for a lattice Ω is

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Omega^*} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$

For z bounded and staying away from the lattice points, $1/(z - \omega)^2 - 1/\omega^2$ has order of magnitude $1/|\omega^3|$. Hence, $\wp(z)$ is meromorphic with a pole of order 2 at each lattice point.

Theorem

6.4

$\wp$ is periodic, hence an elliptic function.

Proof. First, note that $\wp$ is an even function: $\wp(-z) = \wp(z)$, and $\wp'$ is odd. In particular,

$$\wp'(z) = -2 \sum_{\omega \in \Omega} \frac{1}{(z - \omega)^3},$$

which we know is an elliptic function. From the periodicity, we obtain $c \in \mathbb{C}$ such that

$$\wp(z + \omega_1) - \wp(z) = c.$$

Let $z := -\omega_1/2$. Then,

$$\wp(\omega_1/2) = \wp(-\omega_1/2) + c,$$

thus $c = 0$ because p is even. Similarly $\wp(z + \omega_2) = \wp(z)$, so $\wp$ is elliptic. $\square$

Let $\mathcal{M}(\Omega) = \mathcal{M}(T)$ denote the field of all elliptic functions with respect to Ω . Our goal is to classify $\mathcal{M}(\Omega)$. We will show $\mathcal{M}(\Omega) = \mathbb{C}(\wp, \wp')$.

Theorem

6.5

If P is a fundamental parallelogram with $0, \omega_1/2, \omega_2/2$, and $\omega_3/2 := (\omega_1 + \omega_2)/2$ in its interior, then $\omega_1/2, \omega_2/2, \omega_3/2$ are the zeros of $\wp'$ in P .

Proof. Note that $\wp'(z)$ has order 3, hence $\wp'$ has three zeros in P . If $\omega \in \Omega$, we have

$$\wp'(\omega/2) = \wp'(-\omega/2) = -\wp'(\omega/2)$$

since $\omega/2 \equiv -\omega/2 \pmod{\Omega}$. Therefore, $\wp'(\omega/2)$ equals 0 or ∞ . Since the only pole of $\wp'$ is the triple pole at 0, we deduce that $\wp'(\omega/2) = 0$ for $\omega = \omega_1, \omega_2, \omega_3$. $\square$

Remark 6.6. Let $e_i := \wp(\omega_i/2)$ for $i = 1, 2, 3$. Since

$$S := \left[\frac{\omega_1}{2} \right] \cup \left[\frac{\omega_2}{2} \right] \cup \left[\frac{\omega_3}{2} \right]$$

is the set of all zeros of $\wp'$ on $\mathbb{C}$, we see that

$$\{e_1, e_2, e_3\} = \wp(S)$$

is independent of the choice of basis ω_1, ω_2 for Ω .

Corollary**6.7**

For each point $c \in \mathbb{P}^1 \setminus \{e_1, e_2, e_3, \infty\}$, the equation $\wp(z) = c$ has 2 simple solutions, and for $c \in \{e_1, e_2, e_3, \infty\}$, the equation $\wp(z) = c$ has a double solution.

Proof. This is clear, since e_1, e_2, e_3, ∞ are the critical values of the branched cover $\wp : T \rightarrow \mathbb{P}^1$ given by $\wp$, while $0, \omega_1/2, \omega_2/2, \omega_3/2$ are the branch points. $\square$

Theorem**6.8**

If f is an even elliptic function, then $f = R_1(\wp)$ for some rational function $R_1 \in \mathbb{C}(z)$. Thus,

$$\mathcal{M}^{\text{even}}(\Omega) = \mathbb{C}(\wp).$$

Moreover, if f is any elliptic function, then

$$f = R_1(\wp) + \wp' R_2(\wp),$$

where $R_1, R_2 \in \mathbb{C}(z)$. Hence,

$$\mathcal{M}(\Omega) = \mathbb{C}(\wp, \wp').$$

Proof. We begin with the even elliptic functions. Let f be a non-constant even elliptic function of order $N > 0$. If $k \in \mathbb{C}$, the equation $f(z) = k$ has multiple roots where $f'(z) = 0$, and this occurs at only finitely many points of P . Hence, $f(z) = k$ has only simple roots for all but finitely many values of k . Therefore, we can choose $c, d \in \mathbb{C}$ with $c \neq d$ such that

- (a) the roots of $f(z) = c$ and $f(z) = d$ are all simple,
- (b) none of these roots is congruent to 0 or $\omega_i/2$ modulo Ω , $i = 1, 2, 3$.

Since f is even, a complete set of solutions to $f(z) = c$ looks like $A := \{a_1, -a_1, \dots, a_n, -a_n\}$, these being simple and distinct modulo Ω . We say the same for the roots $B := \{b_1, -b_1, \dots, b_n, -b_n\}$ of $f(z) = d$. Therefore,

$$g(z) := \frac{f(z) - c}{f(z) - d}$$

has simple zeros at A , and simple poles at B . By [Corollary 6.7](#), the equation $\wp(z) = \wp(a_i)$ and $\wp(z) = \wp(b_i)$ have simple roots at $z = \pm a_i$ and $z = \pm b_i$ respectively, for $1 \leq i \leq n$. Hence,

$$h(z) := \frac{\prod_{i=1}^n (\wp(z) - \wp(a_i))}{\prod_{i=1}^n (\wp(z) - \wp(b_i))}$$

has the same zeros and poles as $g(z)$ all of them simple. Therefore, by [Corollary 5.6](#), $g = \lambda h$ for some $\lambda \in \mathbb{C}^*$. Now, solve

$$\frac{f(z) - c}{f(z) - d} = \lambda \frac{\prod_{i=1}^n (\wp(z) - \wp(a_i))}{\prod_{i=1}^n (\wp(z) - \wp(b_i))}$$

for $f(z)$, and we get $f(z) \in \mathbb{C}(\wp)$.

Now the general case. If f is odd, the $f/\wp'$ is even, so $f = \wp' R_2(\wp)$ by the even case. In general, for any $f \in \mathcal{M}(\Omega)$, we may write

$$f(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2},$$

and the two terms are even and odd respectively. This concludes the proof. $\square$

We've shown that $\mathcal{M}(\Omega) = \mathbb{C}(\wp, \wp')$. This is not to say that $\wp$ and $\wp'$ are algebraically independent: this would be absurd anyway. We now want to know understand their relationship.

7. The Differential Equation for $\wp(z)$

Definition

7.1

We define the **Weierstrass ζ -function**

$$\zeta(z) := \frac{1}{z} + \sum_{\omega \in \Omega^*} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right). \quad (1)$$

Note that

$$\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} = \frac{z^2}{\omega^2(z - \omega)},$$

so by comparison with

$$\sum_{\omega \in \Omega^*} \frac{1}{|\omega|^3}$$

we see that (1) converges absolutely for $z \in \mathbb{C} \setminus \Omega$.

Let $D := D(0, R)$ be the largest open disc centered at 0 and containing no other lattice points. For $\omega \in \Omega^*$, the series

$$\frac{1}{z - \omega} = -\frac{1}{\omega} - \frac{z}{\omega^2} - \frac{z^2}{\omega^3} - \dots$$

converges absolutely for $z \in D$. We can substitute this in (1) and reverse the order of summation to get

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Omega^*} \left(-\frac{z^2}{\omega^3} - \frac{z^3}{\omega^4} - \dots \right) = \frac{1}{z} - G_3 z^2 - G_4 z^3 - \dots$$

for all $z \in D$, where

$$G_k := G_k(\Omega) := \sum_{\omega \in \Omega^*} \frac{1}{\omega^k}$$

are the **Eisenstein** series for Ω .

For k odd, the terms $1/\omega^k$ and $1/(-\omega)^k$ cancel, so $G_k = 0$. Therefore, the Laurent series for $\zeta(z)$ at 0 is

$$\zeta(z) = \frac{1}{z} - \sum_{n=2}^{\infty} G_{2n} z^{2n-1}$$

for $z \in D$. Expanding, we get

$$\wp(z) = -\zeta'(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \dots,$$

hence

$$\begin{aligned} \wp'(z) &= -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + \dots \\ \wp'(z)^2 &= \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + z^2 \varphi_1(z) \\ 4\wp(z)^3 &= \frac{4}{z^6} + \frac{36G_4}{z^2} + 60G_6 + z^2 \varphi_2(z) \end{aligned}$$

$$60G_4\wp(z) = \frac{60G_4}{z^2} + z^2\varphi_3(z)$$

for $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{O}(0)$. We deduce that

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) + 140G_6 = z^2\varphi(z), \quad (2)$$

where $\varphi(z) = \varphi_1(z) - \varphi_2(z) + \varphi_3(z)$ converges in D . Since $\wp', \wp \in \mathcal{M}(\Omega)$, the LHS in (2) is elliptic, and equals $z^2\varphi(z)$ on D . Hence, the LHS has no poles, so is constant. This constant is zero since $z^2\varphi(z)|_{z=0} = 0$.

Theorem

7.2

The Weierstrass $\wp$ function satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6.$$

Then, writing $g_2 := 60G_4 = 60 \sum_{\omega \neq 0} \frac{1}{\omega^4}$, and $g_3 := 140G_6 = 140 \sum_{\omega \neq 0} \frac{1}{\omega^6}$, we have

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Thus, the points $(\wp(z), \wp'(z))$ lie in the algebraic curve with equation

$$y^2 = 4x^3 - g_2x - g_3,$$

called an **elliptic curve**. The RHS is a cubic with discriminant $\Delta = 16(g_2^3 - 27g_3^2)$.

Proof. Above. □

Moreover, it's possible to show that

$$\wp^{-1}(w) = \int \frac{dw}{\sqrt{4w^3 - g_2w - g_3}}.$$

Thus, the inverses of elliptic functions appear as indefinite integrals.

Proposition

7.3

The discriminant Δ of the elliptic curve above is never 0. Equivalently, the three roots e_1, e_2, e_3 of $\wp'(z) = 0$ are mutually distinct.

Proof. Let $h_j(z) := P(z) - e_j$ for $j = 1, 2, 3$. Then, the poles of h_j are the same as those of $\wp$, and h_j is elliptic of order 2. Hence, h_j has two classes of zeros, counting multiplicities, on the corresponding torus T . But,

$$h_j(\omega_j/2) = h'_j(\omega_j/2) = 0,$$

hence h_j has a double zero at $[\omega_j/2]$, and no more zeros. For $k \neq j$, it follows that

$$e_k - e_j = h_j(\omega_k/2) \neq 0,$$

and we're done. □

Remark 7.4. Let $p(x) := 4x^3 - g_2x - g_3$. Then, $\wp' = \sqrt{p(\wp)}$. Using this, one can construct a homeomorphism $\alpha : \mathbb{C}/\Omega \rightarrow X$, where X is the “Riemann surface of the algebraic function $y^2 = p(x)$ ”. Conversely, one can show that if $p(x)$ is any cubic polynomial $4x^3 - g_2x - g_3$ with distinct roots, then there exists a lattice $\Omega \in \mathbb{C}$ such that the associated Weierstrass function $\wp$ satisfies $\wp'(z)^2 = p(\wp)$.

8. Weierstrass' σ -Function

8.1. Background on Infinite Products

If $\{b_n\}_{n \geq 1}$ is a sequence of non-zero complex numbers, we say that the infinite product $\prod_{n=1}^{\infty} b_n$ converges to $p \in \mathbb{C}$ if $p \neq 0$, and

$$\lim_{N \rightarrow \infty} \prod_{n=1}^N b_n = p.$$

In this case, we must have $\lim_{n \rightarrow \infty} b_n = 1$, so putting $b_n = 1 + c_n$, then $\lim_{n \rightarrow \infty} c_n = 0$.

Since $\lim_{n \rightarrow \infty} b_n = 1$, we may assume $\operatorname{Re} b_n > 0$, and choose a branch Log of $\log z$ defined on $\{\operatorname{Re} z > 0\}$, for example $\operatorname{Log}(z) := \log |z| + i \arg(z)$, where $-\pi < \arg(z) < \pi$. Assuming $b_n \neq 0$ for all n , we can show the following prop

Proposition

8.1

Suppose $b_n \neq 0$ for all n .

1. $\prod_{n=1}^{\infty} b_n$ converges $\iff \sum_{n=1}^{\infty} \operatorname{Log}(b_n)$ converges, in which case

$$\prod_{n=1}^{\infty} b_n = \exp \left(\sum_{n=1}^{\infty} \operatorname{Log}(b_n) \right)$$

2. We say $\prod_{n=1}^{\infty} b_n$ **converges absolutely** if $\sum_{n=1}^{\infty} \operatorname{Log}(b_n)$ converges absolutely. Then, the product $\prod_{n=1}^{\infty} (1 + c_n)$ converges absolutely $\iff \sum_{n=1}^{\infty} |c_n| < \infty$.
3. If $\{f_n\}$ is a sequence of holomorphic functions on an open $U \subset \mathbb{C}$, and if $f_n = 1 + F_n$, we say that the product $\prod_{n=1}^{\infty} f_n$ **converges uniformly on U** if and only if the series $\sum_{n=1}^{\infty} F_n$ converges uniformly on U . Moreover, uniform convergence on compact subsets of U implies $\prod_{n=1}^{\infty} f_n$ is holomorphic on U .

Proof. To prove (2), we use the inequality

$$\frac{|z|}{2} \leq |\operatorname{Log}(1 + z)| \leq 2|z|$$

for small z , which holds since

$$\lim_{z \rightarrow 0} \frac{\operatorname{Log}(1 + z)}{z} = 1.$$

□

Now, we're going to want some terms to be zero in an infinite product.

Definition

8.2

We extend our definition of convergence to mean $\prod_{n=1}^{\infty} b_n$ converges when there exists $N \in \mathbb{N}$ such that $b_n \neq 0$ for all $n \geq N$, and $\prod_{n=N}^{\infty} b_n$ converges to a non-zero complex number p . The value of $\prod_{n=1}^{\infty} b_n$ is then defined to be $b_1 \cdots b_{N-1} \cdot p$.

8.2. The σ -Function**Definition**

8.3

Given a lattice $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, define the **Weierstrass σ -function** to be

$$\sigma(z) := z \prod_{\omega \in \Omega^*} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right).$$

The exponential factor is included to ensure convergence.

Proposition

8.4

$\sigma(z)$ converges.

Proof. Let $K \subset \mathbb{C}$ be any compact set, and for $\omega \neq 0$

$$g_\omega(z) := \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right).$$

Then, as $|\omega| \rightarrow \infty$, we have $g_\omega(z) \rightarrow 1$ converges uniformly on K . Now, for $|\omega|$ large — say $|\omega| > 2|z|$ for $z \in K$ — we have

$$\begin{aligned} |\operatorname{Log} g_\omega(z)| &= \left| \operatorname{Log} \left(1 - \frac{z}{\omega}\right) + \frac{z}{\omega} + \frac{z^2}{2\omega^2} \right| = \left| \frac{1}{3} \frac{z^3}{\omega^3} + \frac{1}{4} \frac{z^4}{\omega^4} + \cdots \right| \\ &\leq \frac{1}{3} \left| \frac{z}{\omega} \right|^3 (1 + 1/2 + 1/4 + \cdots) \leq \left| \frac{z}{\omega} \right|^3 \end{aligned}$$

for all $z \in K$. Then, since $\sum_{\omega \neq 0} \frac{1}{|\omega|^3} < \infty$, we see that

$$z \prod_{\omega \neq 0} g_\omega(z)$$

converges uniformly on compact subsets of $\mathbb{C}$ to $\sigma(z)$. $\square$

Remark 8.5. Observe that $\sigma(-z) = -\sigma(z)$, so σ is an odd function with simple zeros at the points of Ω . Also,

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} = \frac{d}{dz} (\operatorname{Log} \sigma(z)).$$

Since $\zeta'(z) = -\wp(z)$ and $\wp$ is periodic, there is a constant $\eta(\omega)$ such that

$$\zeta(z + \omega) = \zeta(z) + \eta(\omega)$$

for all $\omega \in \Omega$. Clearly, $\eta(\omega)$ is $\mathbb{Z}$ -linear, so if $\eta_1 := \eta(\omega_1)$ and $\eta_2 := \eta(\omega_2)$, then

$$\eta(\omega) = \eta(m\omega_1 + n\omega_2) = m\eta_1 + n\eta_2.$$

Legendre's Relation

8.6

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i.$$

Proof. Since $\zeta(z)$ is meromorphic with a single pole at 0 with residue 1, we get that

$$\int_{\partial P} \zeta(z) dz = 2\pi i. \quad (3)$$

Break ∂P into the four sides $\gamma_1, \dots, \gamma_4$. Then, observe

$$\int_{\gamma_3} \zeta(z) dz = - \int_{\gamma_1} \zeta(z + \omega_w) dz = - \int_{\gamma_1} [\zeta(z) + \eta_2] dz,$$

hence

$$\int_{\gamma_1} \zeta + \int_{\gamma_3} \zeta = - \int_{\gamma_1} \eta_2 dz = -\eta_2 \omega_1.$$

Similarly,

$$\int_{\gamma_2} \zeta + \int_{\gamma_4} \zeta = \eta_1 \omega_2,$$

so plugging into (3) yields the result. $\square$

Now, we examine how $\sigma(z)$ behaves under translation. Using

$$\frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$$

gives

$$\frac{\sigma'(z + \omega)}{\sigma(z + \omega)} = \frac{\sigma'(z)}{\sigma(z)} + \eta(\omega)$$

for all $\omega \in \Omega$. Integrating yields

$$\log \sigma(z + \omega) = \log \sigma(z) + \eta(\omega)z + c$$

where c depends only on ω . Hence,

$$\frac{\sigma(z + \omega)}{\sigma(z)} = e^{\eta(\omega)z + c}. \quad (4)$$

Now evaluating c : assume that $\omega/2 \notin \Omega$, so $\sigma(\omega/2) \neq 0$. Put $z := -\omega/2$ in (4), and we use the fact that σ is odd to obtain

$$\sigma(\omega/2) = -\sigma(\omega/2)e^{-\frac{1}{2}\eta\omega + c} \implies e^c = -e^{\frac{1}{2}\eta\omega},$$

where $\eta := \eta(\omega)$. Hence, if $\omega/2 \notin \Omega$, we have

$$\frac{\sigma(z + \omega)}{\sigma(z)} = -e^{\eta(z + \omega/2)}.$$

On the other hand, for any $\omega \in \Omega$, let

$$\varepsilon(\omega) := \frac{\sigma(z + \omega)}{\sigma(z)} e^{-\eta(z + \omega/2)}$$

so that $\varepsilon(\omega) = -1$ if $\omega/2 \notin \Omega$. Then,

$$\frac{\sigma(z + 2\omega)}{\sigma(z)} = \frac{\sigma(z + 2\omega)}{\sigma(z + \omega)} \frac{\sigma(z + \omega)}{\sigma(z)} \implies \varepsilon(2\omega) = \varepsilon(\omega)^2$$

for all $\omega \in \Omega$. Therefore, if $\omega/2 \in \Omega$, then

$$\varepsilon(\omega) = \varepsilon(\omega/2)^2 = \varepsilon(\omega/4)^4 = \dots = (-1)^{2^n} = 1.$$

Theorem

8.7

We have

$$\frac{\sigma(z + \omega)}{\sigma(z)} = \varepsilon(\omega) \exp \left(\eta(\omega) \cdot \left(z + \frac{\omega}{2} \right) \right),$$

where

$$\varepsilon(\omega) := \begin{cases} 1 & \omega/2 \in \Omega \\ -1 & \omega/2 \notin \Omega. \end{cases}$$

Moreover, if $\omega = m\omega_1 + n\omega_2$, then

$$\varepsilon(\omega) = (-1)^{mn+m+n}.$$

Proof. Above. □

We return to the problem of finding $f \in \mathcal{M}(\Omega)$ with $[a_1], \dots, [a_r]$ and $[b_1], \dots, [b_s]$ as its congruence classes of zeros and poles, with multiplicities $k_1, \dots, k_r$ and $\ell_1, \dots, \ell_s$ respectively. We already have the necessary conditions

- (i) $\sum_{j=1}^r k_j = \sum_{i=1}^s \ell_i$.
- (ii) The sets $[a_1], \dots, [a_r]$ and $[b_1], \dots, [b_s]$ are disjoint.
- (iii) $\sum_{j=1}^r k_j a_j \equiv \sum_{j=1}^s \ell_j b_j \pmod{\Omega}$.

Theorem

8.8

If conditions (i)-(iii) above hold, then there exists an elliptic function, unique up to scalar multiple, with exactly the wanted zeros and poles.

Proof. Let $n = \sum_{j=1}^r k_j = \sum_{i=1}^s \ell_i$. Write $u_1, \dots, u_n$ for the classes $[a_1], \dots, [a_r]$, with a_i taken k_i times, and similarly write $v_1, \dots, v_n$ for $[b_1], \dots, [b_s]$. Then, condition (iii) says

$$\sum_{j=1}^n (u_j - v_j) = \omega \in \Omega,$$

so we may replace u_1 by $u_1 + \omega$, so that

$$\sum_{j=1}^n u_j = \sum_{j=1}^n v_j.$$

Let

$$f(z) := \frac{\sigma(z - u_1) \cdots \sigma(z - u_n)}{\sigma(z - v_1) \cdots \sigma(z - v_n)},$$

where σ is the σ -function associated to the lattice Ω . Note that f is meromorphic on $\mathbb{C}$. Now, since $\omega_1/2 \notin \Omega$, we know

$$\sigma(z - u_j + \omega_1) = -\sigma(z - u_j) \exp(\eta_1 \cdot (z - u_j + \omega_1/2))$$

$$\sigma(z - v_j + \omega_1) = -\sigma(z - v_j) \exp(\eta_1 \cdot (z - v_j + \omega_1/2))$$

for all $j = 1, \dots, n$, and similarly for ω_2 . Therefore, for $i = 1, 2$,

$$f(z + \omega_i) = f(z) \exp\left(\eta_i \sum_{j=1}^n (v_j - u_j)\right) = f(z),$$

and so $f(z) \in \mathcal{M}(\Omega)$. □

Corollary**8.9**

For any $a \in \mathbb{C}$ not in Ω , we have

$$\wp(z) - \wp(a) = -\frac{\sigma(z-a)\sigma(z+a)}{\sigma^2(z)\sigma^2(a)}$$

Proof. The function $\wp(z) - \wp(a)$ has zeros at a and $-a$, and a double pole at 0. Hence, by the theorem and its proof,

$$\wp(z) - \wp(a) = c \frac{\sigma(z-a)\sigma(z+a)}{\sigma^2(z)}$$

for some $c \in \mathbb{C}$. Multiply by z^2 and let $z \rightarrow 0$. Using that $\frac{\sigma^2(z)}{z^2} \rightarrow 1$ and $z^2\wp(z) \rightarrow 1$, we have

$$c = \frac{1}{\sigma(a)\sigma(-a)} = -\frac{1}{\sigma^2(a)}$$

as desired. $\square$

9. Several Complex Variables

Recall the Cauchy-Riemann equations in 1-variable: if $f : U \rightarrow \mathbb{C}$ with is continuous on $U \subset \mathbb{C}$ open, and has all its first partial derivatives, then we set

$$\frac{\partial f}{\partial z}(a) := \frac{1}{2} \left(\frac{\partial f}{\partial x}(a) - i \frac{\partial f}{\partial y}(a) \right) \quad \frac{\partial f}{\partial \bar{z}}(a) := \frac{1}{2} \left(\frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a) \right).$$

Then, f is holomorphic on U if and only if the Cauchy-Riemann equations hold, which we can write as

$$\frac{\partial f}{\partial \bar{z}}(a) = 0, \quad \forall a \in U.$$

Note that size $z = x + iy$ and $\bar{z} = x - iy$, then $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, so the formulas for $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ are just the chain rule.

Now, let Ω be open in $\mathbb{C}^n$.

Definition**9.1**

A function $f : \Omega \rightarrow \mathbb{C}$ is **holomorphic** if for every point $a := (a_1, \dots, a_n) \in \Omega$, there exists a neighborhood U of a and constants $c_\alpha := c_{\alpha_1, \dots, \alpha_n}$, $\alpha_j \in \mathbb{Z}_{\geq 0}$ such that

$$f(z) = \sum_{\alpha} c_{\alpha} (z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha_n}, \quad z \in U. \quad (5)$$

If this series (5) converges for $z = w := (w_1, \dots, w_n)$, then it converges uniformly on compact subsets of the polydisk

$$\{z \mid |z_j - a_j| < |w_j - a_j|, 1 \leq j \leq n\}.$$

One can show that (5) can be differentiated termwise. Hence,

$$c_{\alpha} = \frac{1}{\alpha!} D^{\alpha} f(a),$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$, and $D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

Theorem**9.2**

Let $\Omega \subset \mathbb{C}^n$ be open, and $f : \Omega \rightarrow \mathbb{C}$ be continuous. Suppose that for any $a \in \Omega$, the partial derivatives

$$\frac{\partial f}{\partial \bar{z}_j} := \frac{1}{2} \left(\frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j} \right)$$

exists at a , and $\frac{\partial f}{\partial \bar{z}_j}(a) = 0$ for $j = 1, \dots, n$. Then f is holomorphic on Ω .

Proof. Let $r > 0$ be such that the closed polydisk

$$\bar{P}(a, r) := \prod_{j=1}^n \bar{D}(a_j, r) = \{z \mid |z_j - a_j| < r, 1 \leq j \leq n\}$$

is contained in Ω . For $w_1 \in D(a_1, r)$ and $z_2, \dots, z_n$ fixed, Cauchy's formula gives

$$f(w_1, z_2, \dots, z_n) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z_1, \dots, z_n)}{z_1 - w_1} dz_1,$$

where C_1 is the circle $t \mapsto a_1 + re^{2\pi i t}$, $t \in [0, 1]$. Let C_j be the circle $t \mapsto a_j + re^{2\pi i t}$, and iterate Cauchy's formula to get

$$f(w_1, \dots, w_n) = \frac{1}{(2\pi i)^n} \int_{C_1} \cdots \int_{C_n} \frac{f(z_1, \dots, z_n)}{\prod_{j=1}^n (z_j - w_j)} dz_n \cdots dz_1$$

for $|w_j - a_j| < r$, $1 \leq j \leq n$. Since f is continuous, f is bounded on $\bar{P}(a, r)$. We may expand for each j , and see that

$$-\frac{1}{z_j - w_j} = \frac{1}{(z_j - a_j) - (w_j - a_j)} = \sum_{m=0}^{\infty} \frac{(w_j - a_j)^m}{(z_j - a_j)^{m+1}},$$

which converges uniformly on compact subsets of $D(a_j, r)$. Hence, $f(w) = \sum_{\alpha} c_{\alpha} (w - a)^{\alpha}$ with

$$c_{\alpha} = \frac{1}{(2\pi i)^n} \int_{C_1} \cdots \int_{C_n} \frac{f(z_1, \dots, z_n)}{\prod_{j=1}^n (z_j - a_j)^{\alpha_j+1}} dz_1 \cdots dz_n,$$

so f is holomorphic. □

Properties of Holomorphic Functions on $\mathbb{C}^n$ **9.3**

Holomorphic functions on $\mathbb{C}^n$ satisfy the principle of analytic continuation, the open mapping theorem, Cauchy's formula for a polydisk, Cauchy's inequalities, and theorems of Weierstrass and Montel.

There are also some properties from single complex analysis that fail in several complex variables.

Hartogs' Theorem, 1905**9.4**

If $n \geq 2$, then a holomorphic function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ has no isolated singularities.

Proof. In fact, something stronger is true: Let $\Omega = B_1 \setminus \bar{B}_2$, where B_1, B_2 are open balls with $\bar{B}_2 \subset B_1$. Then, if $f \in \mathcal{O}(\Omega)$ is holomorphic, then there exists unique $F \in \mathcal{O}(B_1)$ such that $F|_{\Omega} = f$. In particular, an isolated singularity $\bar{B}_2 = \{w\}$ can always be filled in. □

The Riemann Mapping Theorem Fails

9.5

There does not exist a holomorphic non-constant f from a ball to a polydisk in $\mathbb{C}^2$. In fact, there's not even a proper map.

10. The Riemann Surface of an Algebraic Function

Why did we discuss several complex variables? Well, one of our main motivators is zero sets of polynomials. If our polynomial has more than one variable (which they usually do), then this induces a surface in several complex variables.

Suppose $p \in \mathbb{C}[x, y]$ is an irreducible polynomial with $\deg_y P = n > 0$, so

$$P(x, y) = a_0(x)y^n + \cdots + a_{n-1}(x)y + a_n(x). \quad (6)$$

Intuitively, the equation $P(x, y) = 0$ defines an algebraic function $y = y(x)$. We want to make this precise.

Implicit Function Theorem

10.1

Let $f(x, y)$ be a holomorphic function of 2 complex variables x and y , defined on the polydisk

$$\{|x| < r_1, |y| < r_2\} \subset \mathbb{C}_2$$

such that $f(0, 0) = 0$ and $\frac{\partial f}{\partial y}(0, 0) \neq 0$. Then, there exists $\varepsilon, \delta > 0$ such that for any $x \in D(0, \varepsilon) = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$, there is a unique solution $y(x)$ to the equation $f(x, y) = 0$ with $|y(x)| < \delta$. The function $x \mapsto y(x)$ is holomorphic.

Proof. Since $\frac{\partial f}{\partial y}(0, 0) \neq 0$, we can choose $\delta > 0$ such that $f(0, y) \neq 0$ for $0 < |y| \leq \delta$. Now, choose $\varepsilon > 0$ such that $f(x, y) \neq 0$ for all $|x| \leq \varepsilon$ and $|y| = \delta$. This is possible, since f is continuous and does not vanish on the compact set $\{0\} \times \{|y| = \delta\}$. By the argument principle, if $|x| < \varepsilon$, then the integral

$$\frac{1}{2\pi i} \int_{|y|=\delta} \frac{f_y(x, y)}{f(x, y)} dy \quad (7)$$

is an integer $n(x)$ equal to the number of zeros of the function $y \mapsto f(x, y)$ in $\{|y| < \delta\}$. Moreover, our initial choice of δ implies $n(0) = 1$. Since $f(x, y) \neq 0$ for $|x| \leq \varepsilon$ and $|y| = \delta$, the integral (7) is a continuous function on x for $|x| < \varepsilon$. Hence, $n(x) = 1$ for $|x| < \varepsilon$. Therefore, there exists a unique zero $y(x)$ of $f(x, y)$ with $|y(x)| < \delta$. Finally, the residue theorem implies that

$$y(x) = \frac{1}{2\pi i} \int_{|y|=\delta} y \frac{\frac{\partial f}{\partial y}(x, y)}{f(x, y)} dy,$$

and it follows that $x \mapsto y(x)$ is a holomorphic map. $\square$

Lemma

10.2

Let $P(x, y) = a_0(x)y^n + \cdots + a_n(x)$, and $a \in \mathbb{C}$ be such that $a_0(a) \neq 0$, and there is no $b \in \mathbb{C}$ such that $P(a, b) = \frac{\partial P}{\partial y}(a, b) = 0$. Then, there exists $\varepsilon > 0$ and n holomorphic functions

$y_1(x), \dots, y_n(x)$ for $x \in D(a, \varepsilon)$ such that

- (i) $y_i(x) \neq y_j(x')$ if $i \neq j$, $x, x' \in D(a, \varepsilon)$, and $P(x, y_j(x)) = 0$ for $x \in D(a, \varepsilon)$, $j = 1, \dots, n$.
- (ii) If $z \in \mathbb{C}$ and $P(x, z) = 0$ for some $x \in D(0, \varepsilon)$, then $z = y_j(x)$ for a unique $j \in [1, n]$.

Proof. Since $\frac{\partial P}{\partial y}(a, b) \neq 0$ for all solutions b of $P(a, b) = 0$, the polynomial $P(a, y)$ has exactly n roots, say $b_1, \dots, b_n$. Take $\varepsilon > 0$ small and for each i , let $y_i(x)$ be the holomorphic function on $D(a, \varepsilon)$ with $y_i(a) = b_i$ and $P(x, y_i(x)) = 0$ (which exists by the implicit function theorem [Theorem 10.1](#)). Then, for small enough $\varepsilon > 0$, the y_i have the sequence properties (the second follows since the equation $P(x, y) = 0$ has $\leq n$ solutions). $\square$

Proposition

10.3

Let $P(x, y)$ be as in (6). Then, there are only finitely many $x \in \mathbb{C}$ such that the equations $P(x, y) = 0$ and $\frac{\partial P}{\partial y}(x, y) = 0$ have a simultaneous solutions $y \in \mathbb{C}$.

Proof. Set $Q := \frac{\partial P}{\partial y}$. By Euclidean division, there exists $k_0 \geq 1$ so that if $b_0(x) = a_0(x)^{k_0}$, we have

$$b_0(x)P = h_0(x, y)Q + Q_1,$$

where $\deg_y Q_1 < \deg_y Q = n - 1$. Similarly, there exists polynomials $b_j(x)$ such that

$$b_1(x)Q = h_1(x, y)Q_1 + Q_2, \quad \deg_y Q_2 < \deg_y Q_1, \dots$$

$$b_r(x)Q_{r-1} = h_r(x, y)Q_r + Q_{r+1} \quad \deg_y Q_{r+1} < \deg_y Q_r.$$

We may assume $\deg_y Q_{r+1} = 0$, otherwise we continue the division process.

Claim 10.4. $Q_{r+1} \not\equiv 0$.

Proof. Indeed, if $Q_{r+1} = 0$, then $Q_r \mid b_r(x)Q_{r-1}$, so if f is an irreducible factor of Q_r with $\deg_y f > 0$, f must divide Q_{r-1} . Iteration shows that $f \mid Q$ and $f \mid P$, contradicting the irreducibility of P . $\square$

Finally, if $(a, b) \in \mathbb{C}^2$ is such that $P(a, b) = 0 = Q(a, b)$, then $Q_{r+1}(a) = 0$. As $Q_{r+1} \not\equiv 0$, we have

$$\{a \in \mathbb{C} \mid \exists b \in \mathbb{C} \text{ with } P(a, b) = 0 = Q(a, b)\} \subset \{a \in \mathbb{C} \mid Q_{r+1}(a) = 0\}$$

is finite. $\square$

Definition

10.5

We write,

$$S_0 := \{x \in \mathbb{C} \mid a_0(x) = 0\},$$

$$S_1 := \left\{x \in \mathbb{C} \mid \exists y \in \mathbb{C}, P(x, y) = \frac{\partial P}{\partial y}(x, y) = 0\right\},$$

$$S := S_0 \cup S_1 \cup \{\infty\}.$$

Then, $S \subset \mathbb{P}^1$ is called the set of **critical points** of x . Note that S is finite by [Proposition 10.3](#).

Proposition**10.6**

Let $P(x, y)$ be as in (6), and assume $a_0(x) \not\equiv 0$. Let

$$V := V(P) := \{(x, y) \in \mathbb{C}^2 \mid P(x, y) = 0\},$$

and $\pi : V \rightarrow \mathbb{C}$ sending $(x, y) \mapsto x$ be the first projection. Then, the map

$$\pi|_{\pi^{-1}(\mathbb{C} \setminus S_0)} : \pi^{-1}(\mathbb{C} \setminus S_0) \rightarrow \mathbb{C} \setminus S_0$$

is proper.

Proof. We use a lemma which gives a bound on the roots of a polynomial in terms of its coefficients.

Lemma**10.7**

Let $c_1, \dots, c_n \in \mathbb{C}$, and suppose $w^n + c_1 w^{n-1} + \dots + c_n = 0$. Then, $|w| < 2 \max_i |c_i|^{1/i}$, unless $c_i = 0$ for all i .

Proof. Let $c := \max_i |c_i|^{1/i} > 0$, and set $z := w/c$. Then, we have $z^n + \frac{c_1}{c} z^{n-1} + \dots + \frac{c_n}{c^n} = 0$. It now suffices to show $|z| < 2$.

Indeed, since $|c_i| \leq c^i$ for all i , we have $|z|^n \leq |z|^{n-1} + \dots + |z| + 1$. If $|z| \geq 2$. Thus, if $|z| \geq 2$, then

$$1 \leq \frac{1}{|z|} + \dots + \frac{1}{|z|^n} \leq \frac{1}{2} + \dots + \frac{1}{2^n} < 1,$$

a contradiction. □

Now, suppose $K \subset \mathbb{C} \setminus S_0$ is compact. Then, there exists $\delta > 0$ such that $|a_0(x)| \geq \delta$, and $|a_i(x)| \leq 1/\delta$ for all $i \geq 1$ and $x \in K$. If $(x, y) \in V$ and $x \in K$, then

$$y^n + \frac{a_1(x)}{a_0(x)} y^{n-1} + \dots + \frac{a_n(x)}{a_0(x)} = 0,$$

and $\left| \frac{a_i(x)}{a_0(x)} \right| \leq 1/\delta^2$ for all i . By [Lemma 10.7](#), $|y| \leq 2 \max_i (1/\delta^{2/i})$, hence $\pi^{-1}(K)$ is bounded. Moreover, $\pi^{-1}(K) = (K \times \mathbb{C}) \cap V$ is closed in $\mathbb{C}^2$, so $\pi^{-1}(K)$ is compact. □

Corollary**10.8**

Let $P(x, y) \in \mathbb{C}[x, y]$ be irreducible, $V := V(P) \subset \mathbb{C}^2$, and $\pi := \text{pr}_1|_V$. Then, for S the set of critical points of x , the map

$$\pi|_{\pi^{-1}(\mathbb{P}^1 \setminus S)} : \pi^{-1}(\mathbb{P}^1 \setminus S) \rightarrow \mathbb{P}^1 \setminus S$$

is a finite sheeted covering map.

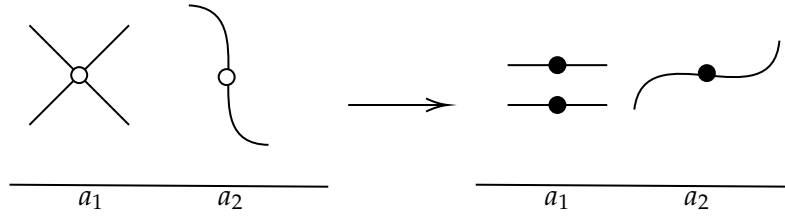
Proof. By [Proposition 10.6](#), π is a proper map. The result then follows from the Implicit Function Theorem ([Theorem 10.1](#)), which gives that π is a local homeomorphism. □

Remark 10.9. The space $Y := \pi^{-1}(\mathbb{P}^1 \setminus S)$ carries the structure of a Riemann surface in a natural way, so that π is holomorphic.

Indeed, if Z is a Riemann surface, X is a connected real 2 dimensional manifold, and $p : X \rightarrow Z$ is a local homeomorphism, then there is a unique complex structure on X such that p is holomorphic.

For U any open set in X such that $p|_U$ is a homeomorphism onto an open set $V \subset Z$ with $V \subset V_j$ for a chart (ψ_j, V_j) of Z , let $\varphi_U : U \rightarrow \mathbb{C}$ be the map $\varphi_U := \psi_j \circ p$. One checks that (U, φ_U) and $(U', \varphi_{U'})$ are compatible, so we get a complex structure on X for which p is holomorphic.

We now complete the space Y to obtain a compact Riemann surface X , with a holomorphic map to $\mathbb{P}^1$ as follows: recall $\pi : Y \rightarrow \mathbb{P}^1 \setminus S$ is a finite covering, with say n sheets. We've removed the problem points S , which can be non-locally Euclidean, or have 0 tangent. Our process will look like this:



Explicitly: Choose a point $a \in S$, and let $\Delta = \Delta_\varepsilon$ be a small disk around a with $\Delta \cap S = \{a\}$. If $a = \infty$, set $\Delta_\varepsilon = \{|z| > 1/\varepsilon\} \cup \{\infty\}$. Then,

$$\pi^{-1}(\Delta^*) = U_1 \cup \cdots \cup U_k$$

has finitely many connected components U_j , $j = 1, \dots, k$. Let $\pi_j := \pi|_{U_j} : U_j \rightarrow \Delta^*$ be a covering map for all j . By the theory of covering spaces, there exists positive integers $m_1, \dots, m_k$ and $\varphi_j : U_j \xrightarrow{\sim} \Delta^*$ such that the diagram

$$\begin{array}{ccc} U_j & \xrightarrow[\sim]{\varphi_j} & \Delta^* \\ \pi_j \downarrow & & \downarrow z \mapsto z^{m_j} \\ \Delta^* & \xrightarrow{\text{id}} & \Delta^* \end{array}$$

commutes for all $j = 1, \dots, k$. If $D \subset \Delta$ is a disk around 0, then $\pi_j^{-1}(D)$ is connected. Now, add a point $P_{j,a}$ to Y for each U_j , and define a neighborhood of $P_{j,a}$ to be $V_{j,a} := U_j \cup \{P_{j,a}\}$, with a map

$$\tilde{\varphi}_j : V_{j,a} \rightarrow \Delta^* \cup \{a\}, \quad P_{j,a} \mapsto 0$$

extending φ_j . Let X be the completed space, and define $p : X \rightarrow \mathbb{P}^1$ by setting $p|_Y = \pi$ and $p(P_{j,a}) = a$ for all j, a . The complex structure on X is determined by the open covering

$$\mathcal{U} := \{V_{j,a}\} \cup \{U \subset Y \mid \pi|_U \text{ is a homeomorphism onto an open set in } \mathbb{C}\}$$

with charts $\varphi_{V_{j,a}} : V_{j,a} \rightarrow \Delta$ the map constructed above, and $\varphi_U : U \rightarrow \mathbb{C}$ being $\pi|_U$. Since $\pi : Y \rightarrow \mathbb{P}^1 \setminus S$ is proper, we deduce that $p : X \rightarrow \mathbb{P}^1$ is proper, so X is compact.

Let $\text{pr}_2 : V \rightarrow \mathbb{C}$ be the second projection, where $V = V(P)$. Then, the function $f := \text{pr}_2|_Y$ is holomorphic.

Claim 10.10. f extends to a meromorphic function on X .

Proof. Let $a \in S, x \in X$ with $p(x) = a$, and choose local coordinates z at x and w at a so that p becomes $z \mapsto z^m = w$. If U is a neighborhood by x , then by definition of V and f , we have

$$f(z)^n + \frac{a_1(w)}{a_0(w)} f(z)^{n-1} + \cdots + \frac{a_n(w)}{a_0(w)} = 0$$

for $z \neq 0, w = p(z)$. Also, the functions $a_i(w)/a_0(w)$ are clearly meromorphic at $w = 0$, so there exists constant $C, N > 0$ such that

$$\left| \frac{a_i(w)}{a_0(w)} \right| \leq \frac{C}{|w|^N}$$

near $w = 0$. Then, by [Lemma 10.7](#),

$$|f(z)| \leq 2 \max_i \frac{C^{1/i}}{|w|^{N/i}} \leq \frac{C_1}{|z|^k}$$

for some constants C_1 and k . Hence, f has a meromorphic extension to all of X . $\square$

Proposition

10.11

Y is connected, so X is as well. Therefore, X is a compact Riemann surface which carries a meromorphic function $f : X \rightarrow \mathbb{P}^1$. Moreover, if $p : X \rightarrow \mathbb{P}^1$ is the extension of $\pi : Y \rightarrow \mathbb{P}^1 \setminus S$, then we have $P(p(x), f(x)) = 0$ on X .

Proof. Suppose Y is not connected, and let Y_1 be a connected component of Y , $Y_1 \neq Y$. Then, $\pi_1 := \pi|_{Y_1} : Y_1 \rightarrow \mathbb{P}^1 \setminus S$ is an r -sheeted covering map for some r with $1 \leq r < n$. Let $f_1 := f|_{Y_1}$. For any $x \in \mathbb{P}^1 \setminus S$, let e_v for $v = 1, \dots, r$ be the v^{th} elementary symmetric function $e_v(x) := e_v(y_1, \dots, y_r)$, where $y_1, \dots, y_r$ are the values taken by f_1 at the r points of $\pi_1^{-1}(x)$. By definition of f_1 , the y_j are values of the 2nd projection pr_2 at the points $(x, y) \in Y$, so we have $P(x, y_j) = 0$ for $1 \leq j \leq r$.

Claim 10.12. *The $e_v(x)$ extend to meromorphic functions on $\mathbb{P}^1$.*

Proof. Indeed, since the y_j are values of the function f_1 , which is meromorphic on Y_1 , in a neighborhood of any $a \in S$, since $\pi_1^{-1}(x) = \{P_1, \dots, P_r\}$, we have an estimate

$$|e_v(x)| = \left| \sum_{i_1 < \dots < i_v} f_1(P_{i_1}) \cdots f_1(P_{i_v}) \right| \leq C \|z\|^{-\ell}.$$

Here, $z = (z_1, \dots, z_r)$, where z_i is a coordinate function near P_i for $i = 1, \dots, r$. Thus,

$$|e_v(x)| \leq C_1 |x - a|^{-\ell'}$$

(or $C_1 |x|^{-\ell}$ if $a = \infty$), proving that the e_v are meromorphic on $\mathbb{P}^1$. $\square$

This claim implies that $e_1(x), \dots, e_r(x)$ are rational functions of x . Let

$$Q(x, y) := y^r - e_1(x)y^{r-1} + \cdots + (-1)^r e_r(x).$$

Then, if $x \in \mathbb{P}^1 \setminus S$, the roots $y_1, \dots, y_r$ of $Q(x, y)$ are also roots of $P(x, y)$. Hence, Q divides P in $\mathbb{C}(x)[y]$, and since $\deg_y Q \geq 1$, Gauss's Lemma implies that P is not irreducible in $\mathbb{C}[x][y]$, a contradiction. $\square$

Definition**10.13**

In algebraic geometry, X is called the **normalization** of the algebraic curve $V = V(P)$

11. The Equation $\partial u / \partial \bar{z} = f$

Let R be a closed rectangle in $\mathbb{C}$, and $f \in C^\infty(R)$ (that is, f is C^∞ in an open neighborhood of R).

Proposition**11.1**

$$\iint_R \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z} = - \int_{\partial R} f dz,$$

where $dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -2i dx \wedge dy$

Proof. Suppose $R = [a, b] \times [c, d]$, and parametrize the four sides of ∂R by $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Then,

$$\begin{aligned} \iint_R \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z} &= -i \int_c^d \int_a^b \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) dx dy \\ &= -i \int_c^d [f(b, y) - f(a, y)] dy \\ &= - \int_{\gamma_2} f dz - \int_{\gamma_4} f dz. \end{aligned}$$

The rest is an exercise. □

Next, let R' and R be closed rectangles with $R' \subset \text{Int}(R)$. Choose f in some closed annulus $C^\infty(R \setminus \text{Int}(R'))$.

Claim 11.2. *We claim*

$$\iint_{R \setminus \text{Int}(R')} \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z} = - \int_{\partial R} f dz + \int_{\partial R'} f dz.$$

Proof. We have $f \in C^\infty(U)$ for some open set $U \supset R \setminus \text{Int}(R')$. Let $\alpha \in C_0^\infty(U)$ with $\alpha \equiv 1$ in a neighborhood of $R \setminus \text{Int}(R')$ (the 0 in the subscript means compactly supported). Consider

$$g(z) := \begin{cases} \alpha(z)f(z) & z \in U \\ 0 & z \notin U. \end{cases}$$

Then, $g \in C^\infty(\mathbb{C})$. Applying [Proposition 11.1](#) to g on R and R' , we obtain

$$\iint_R \frac{\partial g}{\partial \bar{z}} dz \wedge d\bar{z} = - \int_{\partial R} f, \quad \iint_{R'} \frac{\partial g}{\partial \bar{z}} dz \wedge d\bar{z} = - \int_{\partial R'} f,$$

and subtracting gives the claim. □

Theorem**11.3**

Let $f \in C_0^\infty(\mathbb{C})$. Then, for any $\omega \in \mathbb{C}$,

$$f(\omega) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}}(z) \cdot \frac{1}{z - \omega} dz \wedge d\bar{z} \quad (8)$$

Proof. Observe that the function $z \mapsto \frac{1}{z}$ for $0 < |z| \leq 1$ lies in $L^1(\overline{D}(0, 1))$, i.e.

$$\iint_{|z| \leq 1} \frac{1}{|z|} dx dy < \infty.$$

(Indeed, passing to polar coordinates shows that the integral equals 2π). We now change variables in (8) and consider

$$I := \iint_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}}(z + w) \frac{1}{z} dz \wedge d\bar{z}.$$

We claim $I = 2\pi i f(w)$.

Indeed, fix w , and choose a closed rectangle R such that $\text{supp}(z \mapsto f(z + w)) \subset \text{Int}(R)$ and $0 \in R$. Let R_ε be a small square centered at 0 with side length ε , and $R_\varepsilon \subset R$. Using Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \iint_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}}(z + w) \frac{dz \wedge d\bar{z}}{z} &= \iint_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} \left(\frac{f(z + w)}{z} \right) dz \wedge d\bar{z} \\ &= \lim_{\varepsilon \rightarrow 0} \iint_{R \setminus \text{Int}(R_\varepsilon)} \frac{\partial}{\partial \bar{z}} \left(\frac{f(z + w)}{z} \right) dz \wedge d\bar{z} \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\partial R_\varepsilon} \frac{f(z + w)}{z} dz \\ &= 2\pi i f(w) + \lim_{\varepsilon \rightarrow 0} \int_{\partial R_\varepsilon} \frac{f(z + w) - f(w)}{z} dz. \end{aligned}$$

Then, since $f \in C_0^\infty(\mathbb{C})$, there exists $M > 0$ such that $|f(z + w) - f(w)| \leq M|z|$ for $|z|$ small, hence

$$\left| \int_{\partial R_\varepsilon} \frac{f(z + w) - f(w)}{z} dz \right| \leq M \cdot 4\varepsilon \rightarrow 0,$$

proving the claim. $\square$

Theorem

11.4

Let $f \in C_0^\infty(\mathbb{C})$, and define

$$u(w) := \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{f(z)}{z - w} dz \wedge d\bar{z}.$$

Then, $u \in C^\infty(\mathbb{C})$, and $\frac{\partial u}{\partial \bar{z}} = f$ on $\mathbb{C}$.

Proof. We change variables in the integral so that

$$u(w) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{f(z + w)}{z} dz \wedge d\bar{z}.$$

We claim

$$\frac{\partial u}{\partial x}(w) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial f}{\partial x}(z + w)}{z} dz \wedge d\bar{z}.$$

Indeed, for $h \in \mathbb{R}$,

$$\frac{u(w + h) - u(w)}{h} = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{f(z + w + h) - f(z + w)}{h} \cdot \frac{1}{z} dz \wedge d\bar{z}.$$

Since f is differentiable, the first factor in the integral is a bounded function of h , and $\frac{1}{z} dz \wedge d\bar{z}$ is L^1 on any compact set, so we obtain the claim by taking the limit as $h \rightarrow 0$ (the exchange of limit and integral is justified by the dominated convergence theorem).

Arguing in the same way shows that $\frac{\partial u}{\partial \bar{y}}(w)$, and in fact all derivatives of u , exist. Hence, $u \in C^\infty(\mathbb{C})$. Moreover, since $z = x + iy$, and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$, we have

$$\frac{\partial u}{\partial \bar{z}}(w) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}}(z+w) \frac{1}{z} dz \wedge d\bar{z} = f(w)$$

by Theorem 11.3. □

Now, we want to show that $\frac{\partial u}{\partial \bar{z}} = f$ has a solution for any $f \in C^\infty(\Omega)$, i.e. not necessarily compact supported, and any open Ω . For this, we use Runge's theorem.

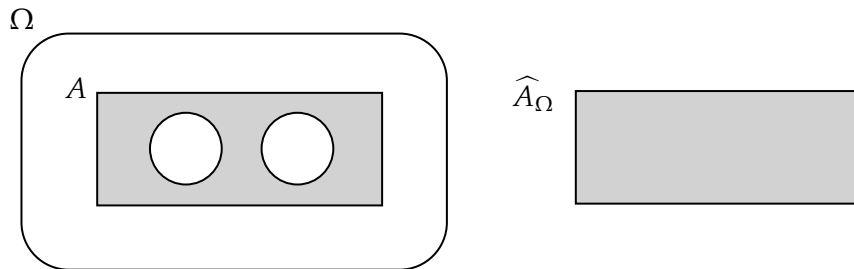
Definition

11.5

Let $\Omega \subset \mathbb{C}$ be open, and $A \subset \Omega$ be any open subset. The **topological hull** $\widehat{A}_\Omega$ of A in Ω is the union of A with the connected components of $\Omega \setminus A$ which are relatively compact (closure is compact) in Ω .

Such a connected component is called a **hole** of A . Intuitively, one gets $\widehat{A}_\Omega$ by filling the holes of A .

Example 11.6. We have the following example:



Exercise 3. If $A \subset \Omega$, then $\widehat{(\widehat{A})}_\Omega = \widehat{A}_\Omega \supset A$. (Don't know what's going on with the TeX here, I'll fix it)

Exercise 4. If $A \subset B \subset \Omega$, then $\widehat{A}_\Omega \subset \widehat{B}_\Omega$.

Proposition

11.7

Let $\Omega \subset \mathbb{C}$ be open and $K \subset \Omega$ be compact. Then, $\widehat{K}_\Omega$ is again compact. Moreover, $\mathbb{C} \setminus \widehat{K}_\Omega$ has only finitely many connected components

Lemma

11.8

Let $\Omega \subset \mathbb{C}$ be open. Then, there exists a sequence $\{K_n\}_{n \geq 1}$ of compact subsets of Ω such that

- (i) $K_n \subset K_{n+1}$,
- (ii) $\bigcup_{n=1}^{\infty} K_n = \Omega$,
- (iii) $K_n = \widehat{K_n}_\Omega$, i.e. $\Omega \setminus K_n$ has no connected components relatively compact to Ω .

Proof. We first construct a sequence $\{K'_n\}$ satisfying the first two conditions.

If $\Omega = \mathbb{C}$, just use disks $K_n = \overline{D}(0, n)$. If $\Omega \neq \mathbb{C}$, then $\partial\Omega \neq \emptyset$, so set

$$K_n = \left\{ x \in \Omega \mid \text{dist}(x, \partial\Omega) \geq \frac{1}{n} \right\} \cap \overline{D}(0, n),$$

and it's easy to show that these satisfy the first two conditions.

Now, set $K_n := (\widehat{K'_n})_\Omega$ for all $n \geq 1$, and we're done. $\square$

Runge's Theorem

11.9

Let $\Omega \subset \mathbb{C}$ be open and $K \subset \Omega$ be compact. Then, any function $f \in \mathcal{O}(\widehat{K}_\Omega)$ can be approximated uniformly on $\widehat{K}_\Omega$, by functions holomorphic on Ω .

Theorem

11.10

Let $\Omega \subset \mathbb{C}$ be open, and $f \in C^\infty(\Omega)$. Then, there exists $u \in C^\infty(\Omega)$ such that $\frac{\partial u}{\partial \bar{z}} = f$.

"The proof is like black magic, and it was by the master of black magic: Grothendieck".

Proof (Grothendieck). Let $\{K_n\}_{n \geq 1}$ be a sequence of compact subsets of Ω as in Lemma 11.8. For each n , there exists $u_n \in C_0^\infty(\Omega)$ such that $\frac{\partial u_n}{\partial \bar{z}} = f$ in a neighborhood of K_n . Then,

$$\frac{\partial}{\partial \bar{z}}(u_{n+1} - u_n) = f - f = 0$$

in a neighborhood of K_n , so $u_{n+1} - u_n \in \mathcal{O}(K_n)$. By Runge's Theorem, there exists $h_n \in \mathcal{O}(\Omega)$ such that

$$\|u_{n+1} - u_n - h_n\|_{K_n} < \frac{1}{2^n}$$

for all $n \geq 1$. Now, define u on K_n by

$$u := u_{n+1} + \sum_{\ell=n}^{\infty} (u_{\ell+1} - u_\ell - h_\ell) - h_1 - h_2 - \cdots - h_{n-1}, \quad (9)$$

and observe that the series converges uniformly on K_n . Moreover, u is well defined on Ω , since

$$\begin{aligned} u &= u_k + \sum_{\ell \geq k} (u_{\ell+1} - u_\ell - h_\ell) - \sum_{j < k} h_j \\ &= u_k + (u_{k+1} - u_k - h_k) + \sum_{\ell \geq k+1} (u_{\ell+1} - u_\ell - h_\ell) - \sum_{j < k} h_j \\ &= u_{k+1} + \sum_{\ell \geq k+1} (u_{\ell+1} - u_\ell - h_\ell) - \sum_{j < k+1} h_j, \end{aligned}$$

i.e. u is independent of the choice of n for K_n . Since the series converges uniformly on K_n , we have

$$u - u_{n+1} \in \mathcal{O}(K_n)$$

and therefore

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u_{n+1}}{\partial \bar{z}} = f$$

on $\text{Int}(K_n)$. Since n is arbitrary, and $\bigcup_n K_n = \Omega$, we're done. $\square$

12. Cohomology and the Mittag-Leffler Problem

The Mittag-Leffler Problem

12.1

Let $\Omega \subset \mathbb{C}$ be open, $E \subset \Omega$ be discrete, and suppose that for all $a \in E$, we're given a principal part

$$p_a := \sum_{v=1}^{\infty} \frac{c_v}{(z-a)^v} \in \mathcal{O}(\mathbb{C} \setminus \{a\})$$

where the c_v are some given constants.

We ask to find a function $f \in \mathcal{O}(\Omega \setminus E)$ such that $f - p_a \in \mathcal{O}(a)$ for all $a \in E$.

Here is a cohomological approach to the problem: Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of Ω by open sets such that $|U_i \cap E| \leq 1$ for all $i \in I$, and let f_i be any meromorphic function that solves the problem on U_i . Note that we can indeed solve the problem on U_i , because if U_i contains one point $a \in E$, we can just take $f = p_a$, and otherwise f can be anything holomorphic.

Now, define

$$f_{ij} := f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j} \in \mathcal{O}(U_i \cap U_j).$$

Then, on $U_i \cap U_j \cap U_k$, we have $f_{ij} + f_{jk} + f_{ki} = 0$. Thus, solving the problem globally on Ω is equivalent to finding a family of functions $\{g_i \in \mathcal{O}(U_i)\}_{i \in I}$ such that $f_{ij} = g_j - g_i$ on $U_i \cap U_j$ for all $i, j \in I$. Indeed, given such a family $\{g_i\}$, we can define $f := f_i + g_i$ on U_i , which is in fact a globally defined function on Ω , because the definition agrees on overlaps. Then, f satisfies the Mittag-Leffler conditions, and conversely a global solution induces such $\{g_i\}$.

In the Čech cohomology theory, $\{f_{ij}\} : f_{ij} + f_{jk} + f_{ki} = 0$ is a cocycle in $Z^1(\mathcal{U}, \mathcal{O})$, and

$$\{\{f_{ij}\} \mid f_{ij} = g_j - g_i \text{ for some } \{g_i\}\} = B^1(\mathcal{U}, \mathcal{O})$$

is a coboundary group. The **first Čech cohomology group**

$$H^1(\mathcal{U}, \mathcal{O}) := Z^1(\mathcal{U}, \mathcal{O}) / B^1(\mathcal{U}, \mathcal{O})$$

is the obstruction to solving the problem in general. The Mittag-Leffler theorem says that this group is zero, for all $\Omega \subset \mathbb{C}$ open, and for all open covers $\mathcal{U}$ of Ω .

13. Sheaves of Abelian Groups

"Sheaves are friends"

Definition

13.1

Let X be a topological space. A **presheaf** $\mathcal{F}$ of abelian groups of X is an assignment $U \mapsto \mathcal{F}(U)$ of abelian groups $\mathcal{F}(U)$ to each open set $U \subset X$, with $\mathcal{F}(\emptyset) = 0$, and for every pair $V \subset U$ of open sets, there are homomorphisms $r_{U,V} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ called **restriction maps**, satisfying

- (i) $r_{U,U} = \text{id}_{\mathcal{F}(U)}$ for all $U \subset X$ open
- (ii) If $W \subset V \subset U$ are three open sets, then $r_{U,W} = r_{V,W} \circ r_{U,V}$.

We usually denote $r_{U,V}(f)$ by $f|_V$, as if f is a function of U and $r_{U,V}(f)$ is its restriction to V .

The presheaf $\mathcal{F}$ is called a **sheaf** if for any $U \subset X$ and open cover $U = \bigcup_{i \in I} U_i$, we have

- (i) If $f, g \in \mathcal{F}(U)$ with $r_{U,U_i}(f) = r_{U,U_i}(g)$ for all $i \in I$, then $f = g$.
- (ii) If we are given, for each $i \in I$, an element $f_i \in \mathcal{F}(U_i)$ such that $r_{U_i, U_i \cap U_j}(f_i) = r_{U_i, U_i \cap U_j}(f_j)$ for all $i, j \in I$, then there exists $f \in \mathcal{F}(U)$ such that $r_{U,U_i}(f) = f_i$ for all $i \in I$.

An element $f \in \mathcal{F}(U)$ is called a **section** of $\mathcal{F}$ over U .

Remark 13.2. By property (i) of a sheaf, the f in (ii) is unique.

Example 13.3. Let X be a Riemann surface.

1. **The sheaf of holomorphic functions** $\mathcal{O}$: for any open set $U \subset X$, $\mathcal{O}(U)$ is the group of holomorphic functions $U \rightarrow \mathbb{C}$
2. **The sheaf of C^∞ -functions** $\mathcal{A}$: for any open set $U \subset X$,

$$\mathcal{A}(U) = C^\infty(U) := \{f : U \rightarrow \mathbb{C} \mid f \text{ is } C^\infty\}$$

3. **The presheaf $\mathcal{F}$ of constant functions**: $\mathcal{F}(U) = \{f : U \rightarrow \mathbb{C} \text{ constant}\}$. Note that $\mathcal{F}$ is not a sheaf, as property (ii) fails: if U is disconnected, we can send one component to 0, another to 1, and then we can't get a constant function that restricts to 0 and 1. However, one gets a sheaf $\mathbb{C}$ by setting $\mathbb{C}(U) := \{f : U \rightarrow \mathbb{C}, f \text{ is locally constant}\}$.

Definition

13.4

Let $\mathcal{F}$ be a presheaf on X , and fix $a \in X$. We define an equivalence relation $\sim$ on pairs (U, f) , where $U \subset X$ is open with $a \in U$, and $f \in \mathcal{F}(U)$ as follows: we say $(U, f) \sim (V, g)$ if there exists W open with $a \in W \subset U \cap V$ such that $f|_W = g|_W$. The equivalence class of a pair (U, f) is called the **germ** of the section f at a , denoted f_a .

The set of all germs of sections at a forms an abelian group $\mathcal{F}_a$ called the **stalk** of $\mathcal{F}$ at a .

Note. We in fact have

$$\mathcal{F}_a := \coprod_{U \ni a} (\mathcal{F}(U) / \sim) =: \varinjlim_{U \ni a} \mathcal{F}(U)$$

We can associate a sheaf to any presheaf $\mathcal{F}$ as follows: Let $|\mathcal{F}| = \coprod_{a \in X} \mathcal{F}_a$ and $p : |\mathcal{F}| \rightarrow X, f_a \mapsto a$ be the natural projection. Define a topology on $|\mathcal{F}|$ as follows: for any $U \subset X$ and $f \in \mathcal{F}(U)$, let $N(U, f) := \{f_a \mid a \in U\}$, where f_a is the equivalence class in $\mathcal{F}_a$ defined by (U, f) . The system of sets $\{N(U, f)\}_{U \subset X}$ is a basis for a topology on $|\mathcal{F}|$ such that p is a local homeomorphism (exercise).

A **section** of $|\mathcal{F}|$ over U is a continuous map $s : U \rightarrow |\mathcal{F}|$ such that $p \circ s = \text{id}_U$, i.e. $s(a) \in \mathcal{F}_a$, for all $a \in U$. Let $|\mathcal{F}|(U)$ denote the set of all sections of $|\mathcal{F}|$ over U .

Exercise 5. $|\mathcal{F}|$, together with the natural restriction maps, is a sheaf.

Exercise 6. For every $a \in X$, there is a natural isomorphism of stalks

$$\mathcal{F}_a \xrightarrow{\sim} |\mathcal{F}|_a.$$

Example 13.5. If X is a Riemann surface, and $p \in X$, we define the *skyscraper sheaf* $\mathbb{C}_p$ by

$$\mathbb{C}_p(U) := \begin{cases} \mathbb{C} & p \in U, \\ 0 & p \notin U. \end{cases}$$

The stalks are

$$(\mathbb{C}_p)_a = \begin{cases} 0 & a \neq p \\ \mathbb{C} & a = p. \end{cases}$$

14. Čech Cohomology of Sheaves

Let $\mathcal{F}$ be a sheaf on the topological space X , and $\mathcal{U} := \{U_i\}_{i \in I}$ be an open cover of X .

Definition

14.1

Define **cochain groups** to be

$$C^0(\mathcal{U}, \mathcal{F}) := \prod_{i \in I} \mathcal{F}(U_i) = \{(f_i)_{i \in I} \mid f_i \in \mathcal{F}(U_i), \forall i \in I\}$$

$$C^1(\mathcal{U}, \mathcal{F}) := \prod_{(i,j)} \mathcal{F}(U_i \cap U_j) = \{(f_{ij})_{i,j \in I} \mid f_{ij} \in \mathcal{F}(U_i \cap U_j), \forall i, j \in I\}$$

Let $Z^1(\mathcal{U}, \mathcal{F}) \subset C^1(\mathcal{U}, \mathcal{F})$ be the subgroup of **1-cocycles**, consisting of families (f_{ij}) for which

$$f_{ij} + f_{jk} = f_{ik} \quad \text{on } U_i \cap U_j \cap U_k, \forall i, j, k \in I \quad (10)$$

Note that (10) implies $f_{ii} = 0$, and $f_{ij} = -f_{ji}$ for any cocycle (f_{ij}) .

Define a **coboundary map** $\delta : C^0(\mathcal{U}, \mathcal{F}) \rightarrow C^1(\mathcal{U}, \mathcal{F})$ by setting $\delta(f_i)_{i,j \in I} := (f_{ij})$, where $f_{ij} := f_j - f_i$ on $U_i \cap U_j$. Moreover, let

$$B^1(\mathcal{U}, \mathcal{F}) := \delta C^0(\mathcal{U}, \mathcal{F}) \subset Z^1(\mathcal{U}, \mathcal{F}).$$

Then, the **1st Čech cohomology group** is

$$H^1(\mathcal{U}, \mathcal{F}) := Z^1(\mathcal{U}, \mathcal{F}) / B^1(\mathcal{U}, \mathcal{F}).$$

Moreover, the **0th cohomology group** is

$$H^0(\mathcal{U}, \mathcal{F}) := \ker \delta \in C^0(\mathcal{U}, \mathcal{F})$$

A 0-cochain (f_i) belongs to $\ker \delta$ if and only if $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j . Since $\mathcal{F}$ is a sheaf, the f_i 's come from the restriction of a section $f \in \mathcal{F}(X)$. Hence, there exists a natural isomorphism

$$H^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(X),$$

so $H^0(\mathcal{U}, \mathcal{F})$ is independent of the open cover $\mathcal{U}$ of X .

Goal. We aim to study, for any open sets $\Omega \subset \mathbb{C}$, the three groups $H^1(\mathcal{U}, \mathcal{A})$, $H^1(\mathcal{U}, \mathcal{O})$, and $H^1(\mathcal{U}, \mathbb{C})$ for any open cover $\mathcal{U}$ of Ω by disks.

14.1. Partitions of Unity

Definition

14.2

A **partition of unity** relative to the open cover $\mathcal{U} := \{U_i\}_{i \in I}$ of $\Omega \subset \mathbb{C}$ is a family $\{\varphi_i\}_{i \in I}$ of C^∞ functions on Ω , such that

- (i) $\varphi_i : \Omega \rightarrow \mathbb{R}_{\geq 0}$ has $\text{supp } \varphi_i \subset U_i$,
- (ii) the family of closed sets $\{\text{supp } \varphi_i\}_{i \in I}$ is **locally finite**, i.e. for any compact set $K \subset \Omega$, the set $\{i \in I \mid K \cap \text{supp } \varphi_i \neq \emptyset\}$ is finite
- (iii) $\sum_{i \in I} \varphi_i = 1$ on Ω .

Partitions of unity always exists for any open cover $\mathcal{U}$ of Ω .

Example 14.3. Let $F \subset \Omega$ be a closed subset and $U \subset \Omega$ be an open set containing F . Then, there exists $\varphi \in C^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$, $\varphi|_F \equiv 1$, and $\varphi|_{\Omega \setminus U} \equiv 0$.

Proof. Set $U^1 := \Omega \setminus F$, and let $\{\varphi, \varphi'\}$ be a partition of unity relative to the cover $\{U, U^1\}$ of Ω . Then, $\varphi'|_F \equiv 0 \implies \varphi|_F \equiv 1$, so φ satisfies the required conditions. $\square$

Theorem

14.4

Let $\Omega \subset \mathbb{C}$ be open and $\mathcal{U} := \{U_i\}_{i \in I}$ be an open cover of Ω . Then,

$$H^1(\mathcal{U}, \mathcal{A}) = 0$$

Proof. Suppose that $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{A})$, so $f_{ij} + f_{jk} = f_{ik}$. We seek a family (f_i) such that $f_{ij} = f_j - f_i$ on $U_i \cap U_j$ for all i, j .

Let $(\varphi_i)_{i \in I}$ be a partition of unity relative to $\mathcal{U}$. Observe that the function $\varphi_i f_{ij}$ can be defined on the set U_j by setting $\varphi_i f_{ij}(z) := 0$ for $z \in U_j \setminus U_i$. Now, define

$$f_j := \sum_{i \in I} \varphi_i f_{ij} \in \mathcal{A}(U_j).$$

Then, for $j, k \in I$, we have on $U_j \cap U_k$ that

$$f_k - f_j = \sum_i \varphi_i (f_{ik} - f_{ij}) = \sum_i \varphi_i (f_{ji} + f_{ik}) = f_{jk} \sum_i \varphi_i = f_{jk}.$$

Hence, $\delta(f_i) = (f_{ij})$. $\square$

Definition

14.5

A sheaf that admits of a partition of unity is called a **fine sheaf**. So, $\mathcal{A}$ is a fine sheaf.

Mittag-Leffler's Theorem, or the 1st Cousin Problem

14.6

Let Ω be open in $\mathbb{C}$, and $\mathcal{U}$ an open cover of Ω . Then,

$$H^1(\mathcal{U}, \mathcal{O}) = 0.$$

Proof. Begin with a 1-cocycle $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{O})$. Let $\alpha_i \in C^\infty(U_i)$ be such that $\alpha_j - \alpha_i = f_{ij}$ on $U_i \cap U_j$ for all i, j . Let $u_i := \partial \alpha_i / \partial \bar{z}$ on U_i . Then, $u_i - u_j = 0$ on $U_i \cap U_j$, so there exists $u \in C^\infty(\Omega)$

such that $u|_{U_i} = u_i$ for all $i \in I$.

Now, by [Theorem 11.10](#), we know there exists $A \in C^\infty(\Omega)$ such that $\frac{\partial A}{\partial \bar{z}} = u$ on Ω . Define (f_i) by $f_i := \alpha_i - A$ on U_i , for all $i \in I$. Then, on $U_i \cap U_j$, $f_j - f_i = \alpha_j - \alpha_i = f_{ij}$, and $\frac{\partial f_i}{\partial \bar{z}} = u_i - u = 0$ on U_i for all $i \in I$. $\square$

Now, we aim to remove our choice of open cover from the definition of the Čech cohomology groups. Namely, we want to define $H^1(X, \mathcal{F})$.

Definition

14.7

Let $\mathcal{U} := \{U_i\}_{i \in I}$ and $\mathcal{V} := \{V_j\}_{j \in J}$ be two open covers of X . Then, $\mathcal{V}$ is said to be a **refinement** of $\mathcal{U}$ if there exists a **refinement map** $\tau : J \rightarrow I$ such that $V_j \subset U_{\tau(j)}$ for all $j \in J$. Such a map τ induces a map

$$\tau^* : H^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(\mathcal{V}, \mathcal{F})$$

sending $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{F})$ to $(g_{\alpha\beta})_{\alpha, \beta \in J}$, where $g_{\alpha\beta} := f_{\tau(\alpha)\tau(\beta)}|_{V_\alpha \cap V_\beta}$.

Check. τ^* depends only on the covering $\mathcal{U}$ and $\mathcal{V}$, and not the refinement map τ .

Lemma

14.8

If $\sigma, \tau : J \rightarrow I$ are two refinement maps, so $V_\beta \subset U_{\sigma(\beta)} \cap U_{\tau(\beta)}$ for all $\beta \in J$, then the induced maps

$$\sigma^*, \tau^* : H^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(\mathcal{V}, \mathcal{F})$$

are equal. Moreover, if $\mathcal{V}$ is a refinement of $\mathcal{U}$, the (now natural) morphism

$$H^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(\mathcal{V}, \mathcal{F})$$

is injective.

Proof. Fix $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{F})$. Then, on $V_{\beta_1} \cap V_{\beta_2}$, we have

$$f_{\tau(\beta_1)\tau(\beta_2)} - f_{\sigma(\beta_1)\sigma(\beta_2)} = g_{\beta_1} - g_{\beta_2},$$

where

$$g_\beta = f_{\tau(\beta)\sigma(\beta)}|_{V_\beta}.$$

Hence, we see that

$$\{f_{\tau(\beta_1)\tau(\beta_2)} - f_{\sigma(\beta_1)\sigma(\beta_2)}\} \in B^1(\mathcal{V}, \mathcal{F}).$$

This proves $\sigma^* = \tau^*$. Injectivity is left as an exercise. $\square$

Suppose we have three coverings $\mathcal{U}, \mathcal{V}, \mathcal{W}$, with $\mathcal{W} < \mathcal{V} < \mathcal{U}$, where $<$ means refinement. Write, in general, $\tau_{\mathcal{U}, \mathcal{V}}$ for the induced map τ^* . Then,

$$\tau_{\mathcal{V}, \mathcal{W}} \circ \tau_{\mathcal{U}, \mathcal{V}} = \tau_{\mathcal{U}, \mathcal{W}}.$$

We can therefore define an equivalence relation σ on the disjoint union $\coprod_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{F})$ over all open coverings by saying $\alpha \in H^1(\mathcal{U}, \mathcal{F})$ is equivalent to $\beta \in H^1(\mathcal{V}, \mathcal{F})$ if there exists a further

refinement $\mathcal{W} < \mathcal{U}$ and $\mathcal{W} < \mathcal{V}$ such that

$$\tau_{\mathcal{U}, \mathcal{W}}(\alpha) = \tau_{\mathcal{V}, \mathcal{W}}(\beta).$$

The set of equivalence classes is in fact a group, and is the direct limit of the $H^1(\mathcal{U}, \mathcal{F})$. We write this as

$$H^1(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{F}) = \coprod_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{F}) / \sim.$$

Then, for any open covering $\mathcal{U}$ of X , the canonical map $H^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(X, \mathcal{F})$ is injective. Thus, $H^1(X, \mathcal{F}) = 0 \iff H^1(\mathcal{U}, \mathcal{F}) = 0$ for all open coverings $\mathcal{U}$ of X .

There is a useful criterion for when an open cover $\mathcal{U}$ of X is sufficient to compute $H^1(X, \mathcal{F})$.

Leray's Theorem

14.9

Let $\mathcal{F}$ be a sheaf of abelian groups on X , and $\mathcal{U} := \{U_i\}_{i \in I}$ be an open cover of X such that $H^1(U_i, \mathcal{F}) = 0$ for all $i \in I$. Then,

$$H^1(X, \mathcal{F}) \simeq H^1(\mathcal{U}, \mathcal{F}).$$

Proof. It suffices to show that for any refinement $\mathcal{V} := \{V_\alpha\}_{\alpha \in A}$ of $\mathcal{U}$, the induced map $\tau^* : H^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(\mathcal{V}, \mathcal{F})$ is surjective. Here, $\tau : A \rightarrow I$ is a refinement map such that $V_\alpha \subset U_{\tau(\alpha)}$ for all $\alpha \in A$.

Let $\{c_{\alpha\beta}\}_{\alpha, \beta \in A} \in Z^1(\mathcal{V}, \mathcal{F})$. Then,

$$\{c_{\alpha\beta}|_{U_i}\} \in Z^1(U_i \cap \mathcal{V}, \mathcal{F}).$$

Since $H^1(U_i, \mathcal{F}) = 0$, there exists a function $g_{i\alpha} \in \mathcal{F}(U_i \cap V_\alpha)$ such that $c_{\alpha\beta} = g_{i\alpha} - g_{i\beta}$ on $U_i \cap V_\alpha \cap V_\beta$. Then, on $U_i \cap U_j \cap V_\alpha \cap V_\beta$, we have

$$g_{i\alpha} - g_{i\beta} = c_{\alpha\beta} = g_{j\alpha} - g_{j\beta} \implies g_{i\alpha} - g_{j\alpha} = g_{i\beta} - g_{j\beta}.$$

Now, since $\mathcal{F}$ is a sheaf, there exists $\gamma_{ij} \in \mathcal{F}(U_i \cap U_j)$ such that $\gamma_{ij} = g_{i\alpha} - g_{j\alpha}$ on $U_i \cap U_j \cap V_\alpha$ for all $\alpha \in A$. Clearly $\gamma_{ij} + \gamma_{jk} = \gamma_{ik}$ on $U_i \cap U_j \cap U_k$, and moreover

$$\gamma_{\tau(\alpha)\tau(\beta)} + c_{\alpha\beta} = (g_{\tau(\alpha)\alpha} - g_{\tau(\beta)\beta}) + (g_{\tau(\beta)\alpha} - g_{\tau(\alpha)\beta}) = -g_{\tau(\beta)\beta} + g_{\tau(\alpha)\alpha}$$

on $V_\alpha \cap V_\beta$, which is a subset of $U_{\tau(\beta)}$. Then, since $g_{\tau(\beta)\beta} \in \mathcal{F}(U_{\tau(\beta)} \cap V_\beta) = \mathcal{F}(V_\beta)$, this proves $(c_{\alpha\beta})$ and $(-\gamma_{\tau(\alpha)\tau(\beta)})$ induce the same element in $H^1(\mathcal{V}, \mathcal{F})$. But, by definition,

$$(-\gamma_{\tau(\alpha)\tau(\beta)}) = \tau^*(-\gamma_{ij}).$$

We conclude that τ^* is surjective, proving the theorem. $\square$

Corollary (Reformulated Mittag-Leffler)

14.10

For any open set $\Omega \subset \mathbb{C}$, we have

$$H^1(\Omega, \mathcal{O}) = 0.$$

14.2. Morphisms on Presheaves

Definition

14.11

A **morphism** of presheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ on X is an assignment, to each open set $U \subset X$, of a map $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ such that for any open sets $V \subset U$, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ r_{U,V} \downarrow & & \downarrow r_{U,V} \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

commutes.

Any morphism φ of sheaves induces a map $\varphi : \mathcal{F}_a \rightarrow \mathcal{G}_a$ of stalks, for all $a \in X$.

Definition

14.12

A sequence

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

of sheaves on X is **exact** if, for all $a \in X$, the induced sequence

$$\mathcal{F}_a \xrightarrow{\alpha} \mathcal{G}_a \xrightarrow{\beta} \mathcal{H}_a$$

is exact.

Example 14.13. Let $X = \mathbb{C}^*$, and for $U \subset X$ open, let $\mathbb{Z}_U$ be the set of locally constant integer valued functions on U , let $\mathcal{O}(U)$ be the set of holomorphic functions on U , and $\mathcal{O}^*(U)$ be the set of nowhere zero holomorphic functions on U .

We claim that the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp(2\pi iz)} \mathcal{O}^* \rightarrow 0$$

is exact.

Proof. For any open set $U \subset X$, the sequence

$$0 \rightarrow \mathbb{Z}(U) \hookrightarrow \mathcal{O}(U) \xrightarrow{\exp(2\pi iz)} \mathcal{O}^*(U)$$

is exact. However, note that $\mathcal{O}(U) \rightarrow \mathcal{O}^*(U)$ need not be surjective: for instance, let $U = X = \mathbb{C}^*$, and take $z \in \mathcal{O}^*(U)$, but we know that an entire logarithm does not exist.

However, for all $z \in \mathbb{C}^*$, $\exp : \mathcal{O}_z \rightarrow \mathcal{O}_z^*$ is surjective: indeed, for $z \neq 0$, the germ of functions near z are holomorphic functions in a small connected domain away from the origin, where we can define a complex logarithm. $\square$

The previous example can be generalized with the following result:

Exercise 7. Use the sheaf axioms to show that for any short exact sequence

$$0 \rightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \rightarrow 0$$

of sheaves on X , the induced sequence

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$$

is exact for any $U \subset X$ open.

Theorem

14.14

Consider a short exact sequence

$$0 \rightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \rightarrow 0 \quad (11)$$

of sheaves on X . Then, the sequence naturally induces a long exact sequence of cohomology groups:

$$0 \rightarrow H^0(X, \mathcal{F}) \xrightarrow{\alpha_X} H^0(X, \mathcal{G}) \xrightarrow{\beta_X} H^0(X, \mathcal{H}) \xrightarrow{\delta} H^1(X, \mathcal{F}) \xrightarrow{\alpha_X^1} H^1(X, \mathcal{G}) \xrightarrow{\beta_X^1} H^1(X, \mathcal{H}).$$

Proof. Since $H^0(X, \mathcal{F}) = \mathcal{F}(X)$, and similar for $\mathcal{G}, \mathcal{H}$, the maps α_X and β_X are already defined. To define α_X^1 , note that $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ induces, for any open cover $\mathcal{U}$ of X , a map

$$\alpha_{\mathcal{U}} : C^1(\mathcal{U}, \mathcal{F}) \rightarrow C^1(\mathcal{U}, \mathcal{G})$$

which sends cocycles to cocycles, and coboundaries to coboundaries. Hence, we obtain a map

$$\alpha_{\mathcal{U}}^1 : H^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(\mathcal{U}, \mathcal{G}).$$

This in turn induces a map α_X^1 , and β_X^1 is defined similarly.

Next, we define the connecting homomorphism $\delta : \mathcal{H}(X) \rightarrow H^1(X, \mathcal{F})$ as follows: let $h \in \mathcal{H}(X)$, and since $\beta : \mathcal{G} \rightarrow \mathcal{H}$ is surjective, there exists an open cover $\mathcal{U} := \{U_i\}_{i \in I}$ of X , and elements $g_i \in \mathcal{G}(U_i)$ such that $\beta_{U_i}(g_i) = h|_{U_i}$ for all $i \in I$. Now,

$$g_i - g_j|_{U_i \cap U_j} \in \ker(\beta_{U_i \cap U_j}) = \text{Im}(\alpha_{U_i \cap U_j}),$$

with the last equality coming from [Exercise 7](#). Hence, there exists $f_{ij} \in \mathcal{F}(U_i \cap U_j)$ with

$$\alpha_{U_i \cap U_j}(f_{ij}) = g_i - g_j|_{U_i \cap U_j}.$$

Clearly, $f_{ij} + f_{jk} + f_{ki}$ maps to zero under $\alpha_{U_i \cap U_j \cap U_k}$. Thus, again by [Exercise 7](#), $f_{ij} + f_{jk} + f_{ki} = 0$ on $U_i \cap U_j \cap U_k$, since the map is injective. Therefore, $\{f_{ij}\} \in Z^1(\mathcal{U}, \mathcal{F})$, so it defines an element $\delta(h)$ in $H^1(X, \mathcal{F})$.

Now, we need to check that $\delta(h)$ is independent of the choices of $\mathcal{U}$ and $\{g_i\}$. Indeed, if $\mathcal{V} := \{V_\alpha\}_{\alpha \in A}$ is a refinement of $\mathcal{U}$ given by the map $\tau : A \rightarrow I$, and we choose elements $g'_\alpha = g_{\tau(\alpha)}|_{V_\alpha}$, we clearly obtain the same element of $H^1(X, \mathcal{F})$. This is just the restriction of the cocycle $\{f_{ij}\}$ defined earlier to $\mathcal{V}$. Hence, we may consider the same covering $\mathcal{U}$, and possibly different $g'_i \in \mathcal{G}(U_i)$ with $\beta_{U_i}(g'_i) = h|_{U_i}$ for all $i \in I$. As above, there exists $f_i \in \mathcal{F}(U_i)$ such that $\alpha_{U_i}(f_i) = g'_i - g_i$. Now, if $f'_{ij} \in \mathcal{F}(U_i \cap U_j)$ maps to $g'_i - g'_j$ on $U_i \cap U_j$, then

$$\alpha_{U_i \cap U_j}(f_i - f_j) = \alpha_{U_i \cap U_j}(f'_{ij} - f_{ij}) \implies f_i - f_j = f'_{ij} - f_{ij}.$$

Hence, $\{f_{ij}\}$ and $\{f'_{ij}\}$ define the same element of $H^1(X, \mathcal{F})$, so δ is well-defined.

Proving exactness of the long exact sequence at $H^0(X, \mathcal{H})$, $H^1(X, \mathcal{F})$, and $H^1(X, \mathcal{G})$ is left as an exercise. $\square$

Question 14.15. When is the differentiation map $\frac{d}{dz} : \mathcal{O}(\Omega) \rightarrow \mathcal{O}(\Omega)$ surjective, for $\Omega \subset \mathbb{C}$ open?

Example 14.16 (The Cohomology Form of Cauchy's Theorem). Let $\Omega \subset \mathbb{C}$ be connected and open. Then, we have an exact sequence of sheaves on Ω :

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{O} \xrightarrow{d/dz} \mathcal{O} \rightarrow 0.$$

Then, since $H^1(\Omega, \mathcal{O}) = 0$ by Mittag-Leffler (Corollary 14.10), we get an induced long exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{O}(\Omega) \rightarrow \mathcal{O}(\Omega) \rightarrow H^1(\Omega, \mathbb{C}) \rightarrow 0$$

by Theorem 14.14. Thus, $H^1(\Omega, \mathbb{C})$ measures the obstruction to solving the equation $f' = g$ for $g \in \mathcal{O}(\Omega)$. If Ω is simply connected, then d/dz is onto and $H^1(\Omega, \mathbb{C}) = 0$.

In fact, one can show

$$H^1(\Omega, \mathbb{C}) \simeq H_{\text{sing}}^1(\Omega, \mathbb{C}),$$

where the RHS is singular cohomology.

Remark 14.17. See the textbook for a definition of higher cohomology groups $H^k(X, \mathcal{F})$ for any $k \geq 0$. One can show that the long exact sequence of cohomology groups extends further:

$$\begin{aligned} 0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{G}(X) \rightarrow \mathcal{H}(X) \rightarrow H^1(X, \mathcal{F}) \rightarrow H^1(X, \mathcal{G}) \rightarrow H^1(X, \mathcal{H}) \\ \rightarrow H^2(X, \mathcal{F}) \rightarrow H^2(X, \mathcal{G}) \rightarrow H^2(X, \mathcal{H}) \rightarrow H^3(X, \mathcal{F}) \rightarrow \dots \end{aligned}$$

15. Vector Bundles, Line Bundles, and Divisors

Let X, E be topological spaces, and $p : E \rightarrow X$ a continuous map. We assume that for every $x \in X$, the fiber $E_x := p^{-1}(x)$ has the structure of an n -dimensional complex vector space.

Definition

15.1

The map $p : E \rightarrow X$ is called a **vector bundle** of rank n over X if every point $a \in X$ has a neighborhood $U \ni a$ such that there is a homeomorphism $\varphi : p^{-1}(U) \rightarrow U \times \mathbb{C}^n$, called a **local trivialization** of E over U , which is

(i) **fiber preserving**: the diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbb{C}^n \\ & \searrow p & \swarrow \text{pr}_1 \\ & U & \end{array}$$

commutes, and

(ii) for each $x \in U$ the map $\varphi|_{E_x}$ is a vector space isomorphism

$$E_x \xrightarrow{\sim} \{x\} \times \mathbb{C}^n \equiv \mathbb{C}^n.$$

An **atlas** for E is a family $\{(U_i, \varphi_i)\}_{i \in I}$ of local trivializations such that $\bigcup_{i \in I} U_i = X$.

A vector bundle of rank 1 is called a **line bundle**.

Definition

15.2

A vector bundle $E \rightarrow X$ is said to be **trivial** if it has a linear trivialization $E \xrightarrow{\sim} X \times \mathbb{C}^n$. Hence, any vector bundle is locally trivial by definition.

Proposition

15.3

Suppose that $E \xrightarrow{p} X$ is a rank n vector bundle over a topological space X , and suppose

$$\varphi_i : p^{-1}(U_i) \cong E_{U_i} \rightarrow U_i \times \mathbb{C}^n$$

is an atlas on E . Then, there exists unique continuous maps

$$g_{ij} : U_i \cap U_j \rightarrow \mathrm{GL}_n(\mathbb{C}) \subset \mathbb{C}^{n^2} \cong M_n(\mathbb{C}).$$

such that the morphisms $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^n \rightarrow (U_i \cap U_j) \times \mathbb{C}^n$ satisfy

$$\varphi_{ij}(x, t) = (x, g_{ij}(x)t), \quad \forall (x, t) \in U_i \cap U_j \times \mathbb{C}^n.$$

On $U_i \cap U_j \cap U_k$, the equation $g_{ij}g_{jk} = g_{ik}$ holds. The $\{g_{ij}\}$ are called **transition functions** for the vector bundle.

Proof. The map $\varphi_i \circ \varphi_j^{-1}$ is a fiber preserving homeomorphism which, when restricted to each fiber, is a vector space isomorphism. Hence, for all $x \in U_i \cap U_j$, there exists a matrix $g_{ij}(x) \in \mathrm{GL}_n(\mathbb{C})$ with $\varphi_{ij}(x, t) = (x, g_{ij}(x)t)$. Since φ_{ij} is a homeomorphism, the map $x \mapsto g_{ij}(x)$ is continuous on $U_i \cap U_j$. $\square$

Remark 15.4. The converse of Proposition 15.3 also holds. Namely, given a family $\{g_{ij}\}$ of continuous maps $U_i \cap U_j \rightarrow \mathrm{GL}_n \mathbb{C}$ such that $g_{ij}g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$, we can construct a vector bundle with these transition functions.

Proof. Set $E' := \coprod_{i \in I} U_i \times \mathbb{C}^n \times \{i\} \subset X \times \mathbb{C}^n \times I$, with the topology induced from $X \times \mathbb{C}^n \times I$, where I has the discrete topology. Let $(x, t, i) \sim (x', t', j)$ if and only if $x = x'$, and $t = g_{ij}(x)t'$. Then, the cocycle condition $g_{ij}g_{jk} = g_{ik}$ implies that $\sim$ is an equivalence relation on E' . Define $E := E' / \sim$, with the projection $p : E \rightarrow X$. If $h : E' \rightarrow E$ is the natural quotient map $h|_{U_i \times \mathbb{C}^n \times \{i\}} \rightarrow E_{U_i}$ is a homeomorphism, and we define φ_i to be the inverse of $h|_{U_i \times \mathbb{C}^n \times \{i\}}$. $\square$

Definition

15.5

Let X be a Riemann surface (or more generally a complex manifold), and $E \rightarrow X$ be a rank n vector bundle over X . Two local trivializations (U_i, φ_i) and (U_j, φ_j) are said to be **compatible** if the associated transition function $g_{ij} : U_i \cap U_j \rightarrow \mathrm{GL}_n(\mathbb{C})$ is holomorphic.

Remark 15.6. If $\Omega \subset \mathbb{C}^m$ is open, then a map $f : \Omega \rightarrow \mathbb{C}^n$ is **holomorphic** if $f = (f_1, \dots, f_n)$, with each $f_i : \Omega \rightarrow \mathbb{C}$ holomorphic.

If X has a covering by a compatible trivializations for E , in which case we say E has a **holomorphic atlas**, then E is called a **holomorphic vector bundle** over X .

Remark 15.7. If $L \rightarrow X$ is a holomorphic line bundle with transition functions $\{g_{ij}\}$ with respect to an

open cover $\mathcal{U} := \{U_i\}_{i \in I}$, then we obtain an associated element of $H^1(X, \mathcal{O}^*)$. Indeed, $\mathrm{GL}_1 \mathbb{C} \equiv \mathbb{C}^*$, and $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ satisfy the cocycle condition on $U_i \cap U_j \cap U_k$.

Definition**15.8**

Let $E \xrightarrow{p} X$ and $E' \xrightarrow{p'} X$ be two vector bundles over a Riemann surface X . A **morphism** $u : E \rightarrow E'$ is a map such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ & \searrow p \quad \swarrow p' & \\ & X & \end{array}$$

commutes (i.e. $p'u = p$), and $u : p^{-1}(x) \rightarrow (p')^{-1}(x)$ is $\mathbb{C}$ -linear for all $x \in X$.

The morphism u is called continuous/ C^∞ /holomorphic if the function u has this property. We use this notion to define **isomorphic vector bundles**.

Example 15.9. E is trivial if and only if E is isomorphic to the trivial vector bundle $X \times \mathbb{C}^n$.

16. Sections of Vector Bundles

Definition**16.1**

If $p : E \rightarrow X$ is a vector bundle, a **section** E is a morphism $s : X \rightarrow E$ such that $s(x) \in E_x$ for all $x \in X$. Such sections can be continuous/ C^∞ /holomorphic as before.

If $\varphi_i : p^{-1}(U_i) \rightarrow U_i \times \mathbb{C}^n$ are local trivialization maps for E , then $\varphi_i \circ s(x) = (x, f_i(x))$ for all $x \in U_i$, where $f_i : U_i \rightarrow \mathbb{C}^n$ is a continuous/ C^∞ /holomorphic map.

On $U_i \cap U_j$, we have

$$\begin{aligned} \varphi_i \circ s(x) &= (x, f_i(x)) = \varphi_i \circ \varphi_j^{-1}(x, f_j(x)) = (x, g_{ij}(x)f_j(x)) \\ \implies f_i(x) &= g_{ij}(x)f_j(x), \quad \forall x \in U_i \cap U_j. \end{aligned} \tag{12}$$

Hence, a section s of E may be identified with a family $\{f_i\}_{i \in I}$ of functions $f_i : U_i \rightarrow \mathbb{C}^n$ such that (12) holds for all $i, j \in I$. The sections of E over any open set $U \subset X$ are defined analogously.

Example 16.2. A section of the trivial vector bundle $E \rightarrow X$ is just a function $f : X \rightarrow \mathbb{C}^n$.

Definition**16.3**

Let X be a Riemann surface and $\pi : E \rightarrow X$ be a holomorphic vector bundle on X . Let $S \subset X$ be a discrete set, and $s : X \setminus S \rightarrow E$ be a holomorphic section of E over $X \setminus S$. Then, s is called a **meromorphic section** if, for all $a \in S$, there exists a neighborhood U of a and a coordinate z on U with $z(a) = 0$ such that $U \cap S = \{a\}$, and for some $N > 0$, $z^N s$ is the restriction to $U \setminus \{a\}$ of a holomorphic section of E over U .

Definition

16.4

The **sheaf of holomorphic sections** $\mathcal{E}$ of E is defined by the group

$$\mathcal{E}(U) = \Gamma(U, E) := \{\text{holomorphic sections of } E \text{ over } U\}.$$

17. Divisors and Line Bundles

Definition

17.1

Let X be a Riemann surface. A **divisor** on X is a map $D : X \rightarrow \mathbb{Z}$ such that the support of D is locally finite. That is, for every compact $K \subset X$,

$$\{p \in X \mid D(p) \neq 0\} \cap K \text{ is finite.}$$

We assume now that X is compact so the support of D is finite.

Definition

17.2

The sum and difference $D_1 + D_2$ and $D_1 - D_2$ of two divisors are defined in the obvious way.

A divisor D is said to be **effective** if $D(p) \geq 0$ for all $p \in X$. We write $D_1 \geq D_2$ if $D_1 - D_2$ is effective. The set $\{p \in X \mid D(p) \neq 0\}$ is called the **support** of D , denoted $|D|$ or $\text{supp}(D)$.

Recall 17.3. If $f : U \rightarrow \mathbb{P}^1$ is a meromorphic function on a neighborhood U of a point $a \in X$, and z is a local coordinate on U with $z(a) = 0$, we set

$$\text{ord}_a(f) := \begin{cases} \infty & f \equiv 0 \text{ near } a, \\ k & f(z) = \sum_{n=k}^{\infty} a_n z^n, a_k \neq 0. \end{cases}$$

Suppose now that s is a meromorphic section of a holomorphic vector bundle E on a Riemann surface X , with $s \not\equiv 0$. For $a \in X$, choose a coordinate neighborhood (U, z) with $z(a) = 0$, and a local trivialization $\varphi : E|_U \rightarrow U \times \mathbb{C}^n$. Then, $(\varphi \circ s)(x) = (x, f(x))$, $x \in U$, where $f = (f_1, \dots, f_n)$, is an n -tuple of meromorphic functions on U . There is a unique integer k such that $f = z^k g$, where $g(a) \neq 0$, and $g = (g_1, \dots, g_n)$ with each g_i holomorphic near a .

Definition

17.4

We define $\text{ord}_a(s) := k$, where k is the constant above.

Exercise 8. The definition of $\text{ord}_a(s)$ is well-defined, i.e. independent of the choice of chart and local trivialization.

Definition

17.5

The **divisor** of s is the function $a \mapsto \text{ord}_a(s)$. That is,

$$\text{div}(s) = \sum_{a \in X} \text{ord}_a(s)[a].$$

Note that this also defines the divisor $\text{div}(f)$ of a meromorphic function f on X , since any such f is a meromorphic section of the trivial line bundle $X \times \mathbb{C} \rightarrow X$.

Lemma**17.6**

Let $D := \sum n_p [p]$ be a divisor on a compact Riemann surface X . Then, $D = \operatorname{div}(s)$ for a meromorphic section s of a line bundle on X , denoted $L(D)$.

Proof. To construct $L(D)$ and s from D , let $S := |D|$ and (U_P, z_P) be a local coordinate chart at $P \in S$ with $z_P(P) = 0$. Assume that $U_P \cap U_Q = \emptyset$ for $P \neq Q$ in S . Set $U_0 := X \setminus S$, $f_0 \equiv 1$ on U_0 , and let $f_P := z_P^{n_P}$ on U_P , for all $P \in S$. Define $I := \{0\} \sqcup S$, and let $g_{ij} := f_i / f_j$ for all $i, j \in I$, with g_{ij} defined on $U_i \cap U_j$. Then, for i, j with $U_i \cap U_j \neq \emptyset$, $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$, and $g_{ij}g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. So, the $\{g_{ij}\}$ are the transition functions for a line bundle $L(D)$ on X . Since $f_i = g_{ij}f_j$ on $U_i \cap U_j$, the f_i 's define the meromorphic section s_D of $L(D)$.

We now have $\operatorname{ord}_a(s_D) = \operatorname{ord}_a(f_0) = 0$ if $a \in X \setminus S$, and $\operatorname{ord}_a(s_D) = \operatorname{ord}_P(z_P^{n_P}) = n_P$, if $a = P \in S$. Hence, $\operatorname{div}(s_D) = D$. $\square$

Overview

Let $L \rightarrow X$ be a line bundle on a Riemann surface, and $\{U_i\}_{i \in I}$ is a trivializing open cover of X . Then, we can identify L with a family of transition functions $\{g_{ij}\}$, with $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$, and $g_{ij}g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. Moreover, if $s : X \rightarrow L$ is a section, we can identify s with $\{f_i\}$ such that f_i is a function on U_i , and $f_i = g_{ij}f_j$ on $U_i \cap U_j$.

Remark 17.7. Let $D = \sum n_p [p]$ be a divisor on X . This associates to $L(D)$, s_D such that $\operatorname{div}(s_D) = D$. The line bundle $L(D)$ is unique up to isomorphism.

Proof. If (U_P, ζ_P) are different local coordinate charts, then $h_P := (\zeta_P / z_P)^{n_P} \in \mathcal{O}^*(U_P)$ and $h_0 := 1$ implies

$$g_{ij}^{\text{new}} = h_i g_{ij}^{\text{old}} h_j^{-1}.$$

 $\square$ **Definition****17.8**

The section s_D is called the **standard section** of $L(D)$.

Remarks 17.9. 1. If s is a meromorphic section of a vector bundle $E \rightarrow X$, $s \not\equiv 0$, then s is holomorphic if and only if $\operatorname{div}(s)$ is effective (so s may have zeros, but no poles).

2. If D is a divisor on X , and $U \subset X$ is open, we obtain a sheaf structure on X by defining

$$\mathcal{O}_D(U) := \{f \text{ meromorphic on } U \mid \operatorname{div}(f) \geq -D\}$$

Moreover, if $\Gamma(U, L(D))$ denotes the vector space of holomorphic sections of $L(D)$ over U , and s_D is the standard section of $L(D)$ with $\operatorname{div}(s_D) = D$, then the map

$$\mathcal{O}_D(U) \rightarrow \Gamma(U, L(D))$$

$$f \mapsto f s_D$$

is an isomorphism. Hence the sheaf of holomorphic sections of $L(D)$ can be canonically identified with $\mathcal{O}_D$.

3. If $L \rightarrow X$ is any holomorphic line bundle, and s_0, s_1 are two meromorphic sections of L , $s_0 \not\equiv 0$, then there exists a meromorphic function f on X with $s_1 = fs_0$. Indeed, s_1/s_0 is well-defined independent of any local trivialization, since $f_i^1 = g_{ij}f_j^1$ and $f_i^0 = g_{ij}f_j^0$ on $U_i \cap U_j$, so $f_i^1/f_j^1 = f_i^0/f_j^0$.

18. Linear Equivalence of Divisors

Definition

18.1

Two divisors D_1 and D_2 are **linearly equivalent** if there exists a meromorphic function f on X , not identically zero, such that $\operatorname{div} f = D_1 - D_2$. We write $D_1 \sim D_2$ when this is the case.

Proposition

18.2

We have $D_1 \sim D_2$ if and only if the line bundles $L(D_1)$ and $L(D_2)$ are holomorphically isomorphic.

Proof. Suppose $D_1 - D_2 = \operatorname{div} f$ for some $f \in \mathcal{M}(X)^*$. Let s_{D_1} and s_{D_2} be the standard sections of $L(D_1)$ and $L(D_2)$.

Claim 18.3. We claim that the map $u := s_{D_1} \mapsto s_{D_2} : L(D_1) \rightarrow L(D_2)$ is a unique isomorphism.

Proof. Indeed first define u outside $|\operatorname{div} f| \cup |D_1| \cup |D_2|$ by

$$\lambda s_{D_1}(x) \mapsto \lambda f(x) s_{D_2}(x), \quad \lambda \in \mathbb{C}.$$

Then, u extends holomorphically to all of X , because on any open set $U \subset X$, the section $x \mapsto \lambda(x) s_{D_1}(x)$ is holomorphic if and only if the section $x \mapsto \lambda(x) f(x) s_{D_2}(x)$ is holomorphic (since $\operatorname{div}(s_{D_1}) \cap U = \operatorname{div}(fs_{D_2}) \cap U$). $\square$

Conversely, if $u : L(D_1) \rightarrow L(D_2)$ is an isomorphism, then $u \circ s_{D_1}$ is a meromorphic section of $L(D_2)$. If f is defined by $u \circ s_{D_1} = fs_{D_2}$, then $\operatorname{div} f = D_1 - D_2$ since $\operatorname{ord}_a(s_{D_1}) = \operatorname{ord}_a(u \circ s_{D_1})$ for all $a \in X$. $\square$

Remark 18.4. In terms of the sheaves $\mathcal{O}_{D_1}$ and $\mathcal{O}_{D_2}$, the above isomorphism is just the map $\mathcal{O}_{D_1}(U) \rightarrow \mathcal{O}_{D_2}(U)$ sending $\varphi \mapsto f\varphi$.

19. Linear Operations on Vector Bundles

Linear operations on vector spaces/matrices can be bundle-ized, provided that they are functorial. This means that they extend to give operations on vector bundles/transition functions.

Example 19.1. Suppose $E \rightarrow X$ and $E' \rightarrow X$ are two vector bundles over the same base X , and $\{g_{ij}\}$ and $\{g'_{ij}\}$ are transition functions for E and E' with respect to the same trivializing open cover of X .

1. **Direct sum:** The direct sum $E \oplus E'$ of E and E' is a vector bundle on X such that $(E \oplus E')_x =$

$E_x \oplus E'_x$, and the transition functions $\{g_{ij}^{E \oplus E'}\}$ are given by

$$g_{ij}^{E \oplus E'} = g_{ij} \oplus g'_{ij} = \begin{pmatrix} g_{ij} & 0 \\ 0 & g'_{ij} \end{pmatrix}$$

2. **Tensor product:** The tensor product $E \otimes E'$ has $(E \otimes E')_x = E_x \otimes E'_x$ and $g_{ij}^{E \otimes E'} = g_{ij} \otimes g'_{ij}$. In particular, if L and L' are two line bundles on X , then $L \otimes L'$ has transition functions $\{g_{ij} \cdot g'_{ij}\}$. Observe that this is the group operation on $H^1(X, \mathcal{O}^*)$.
3. **Dual vector bundle E^* :** If $E \xrightarrow{p} X$ is a holomorphic vector bundle, and $E^* := \coprod_{a \in X} E_a^*$, we make E^* into a vector bundle as follows: for a trivialization $\varphi_U : p^{-1}(U) \rightarrow U \times \mathbb{C}^n$ of E over $U \subset X$, define $\tilde{\varphi}_U : \coprod_{a \in U} E_a^* \rightarrow U \times \mathbb{C}^n$ by

$$\tilde{\varphi}_U(v) = (a, (\varphi_{U,a})^{-1}(v)), \quad v \in E_a^*.$$

It follows that if $\{g_{ij}\}$ are the transition functions of E with respect to an open cover, then the transition functions for E^* are $\{(g_{ij}^t)^{-1}\}$.

4. **Exterior and symmetric powers:** We have the k 'th symmetric power $S^k E$, and the k 'th exterior power $\wedge^k E$, with transition functions $\{S^k g_{ij}\}, \{\wedge^k g_{ij}\}$ respectively.
5. **Hom bundle:** We have the vector bundle $\text{Hom}(E, E') \rightarrow X$, defined via the isomorphism $\text{Hom}(E, E') \simeq E^* \otimes E'$.

Exercise 9. Using local trivializations, show how a global section of $\text{Hom}(E, E')$ can be identified with a morphism $E \rightarrow E'$ of vector bundles.

Remark 19.2. For divisors D, D' , we have

$$\begin{aligned} L(D) \otimes L(D') &\simeq L(D + D'), \\ L(D)^* &\equiv L(-D). \end{aligned}$$

Lemma

19.3

A line bundle $L \rightarrow X$ is trivial if and only if L has a holomorphic section s with $s(x) \neq 0$, for all $x \in X$.

Proof. The map $X \times \mathbb{C} \rightarrow L$ sending $(x, \lambda) \mapsto \lambda s(x)$ is an isomorphism. $\square$

Corollary

19.4

If $L \rightarrow X$ is any line bundle, then $L \otimes L^*$ is a trivial line bundle on X .

Proof. If $v \in L_x, v \neq 0$, then there exists a unique $\ell \in L_x^*$ such that $\ell(v) = 1$. Then, $x \mapsto v \otimes \ell$ is a nowhere vanishing section of $L \otimes L^*$ (indeed, the linear form corresponding to $c v$ is $\frac{1}{c} \ell$ for $c \in \mathbb{C}^*$). $\square$

Remark 19.5. Let L be a holomorphic line bundle on X , and s be a meromorphic section of $L, s \neq 0$. Then $L \simeq L(D)$, where $D = \text{div}(s)$.

Proof. If s_{-D} is the standard section of $L(-D)$, then $s \otimes s_{-D}$ is a nowhere vanishing holomorphic section of $L \otimes L(-D)$. Hence, $L \otimes L(-D) \simeq \mathcal{O}$ (the trivial line bundle), so $L \simeq \mathcal{O} \otimes L(D) \simeq L(D)$.

Alternatively, note that the isomorphism $\lambda s(x) \mapsto \lambda s_D(x)$, which is defined outside $|D|$, extends to an isomorphism $L \rightarrow L(D)$. $\square$

Definition**19.6**

Let $Z^1(X)$ denote the abelian group of all divisors on X (sometimes denoted $\text{Div}(X)$), and let $A^1(X) := Z^1(X) / \sim$, where $\sim$ denotes linear equivalence. Let $\text{Pic}(X)$ be the group of isomorphism classes of line bundles on X , under the tensor product, called the **Picard group**.

Remark 19.7 (Homework exercise). *There is a natural isomorphism*

$$\begin{aligned} \text{Pic}(X) &\xrightarrow{\sim} H^1(X, \mathcal{O}^*) \\ [L] &\mapsto \{g_{ij}\} \end{aligned}$$

where $\{g_{ij}\}$ are the transition functions of L with respect to a trivializing open cover of X .

Remark 19.8. *There is a morphism*

$$\begin{aligned} \varphi : Z^1(X) &\rightarrow \text{Pic}(X) \\ D &\mapsto [L(D)]. \end{aligned}$$

This passes to linear equivalence, so we get an induced map

$$\tilde{\varphi} : A^1(X) \rightarrow \text{Pic}(X).$$

Theorem**19.9**

The map $\tilde{\varphi}$ above is surjective, so $\text{Pic}(X) \simeq A^1(X)$ for X a compact Riemann surface.

Proof. The proof of the claim relies on proving that any holomorphic line bundle $L \rightarrow X$ has a not identically zero meromorphic section. $\square$

20. Finiteness Theorems

We now want to prove some facts about finiteness in cohomology. We first need some facts from complex analysis.

Weierstrass's Theorem**20.1**

Let $\Omega \subset \mathbb{C}$ be an open set and $\{f_n\}_{n \geq 1}$ be a sequence of holomorphic functions on Ω . Suppose $\{f_n\}$ converge uniformly on compact subsets of Ω to a function $f : \Omega \rightarrow \mathbb{C}$. Then, $f \in \mathcal{O}(\Omega)$, and $f'_n \rightarrow f'$ uniformly on compact subsets of Ω .

Montel's Theorem**20.2**

Let $\Omega \subset \mathbb{C}$ be open, and $\mathcal{F} \subset \mathcal{O}(\Omega)$ be a bounded family, i.e. for any compact set $K \subset \Omega$, there exists $M = M_K$ such that $|f(z)| \leq M$, for all $z \in K$ and $f \in \mathcal{F}$. Then, every sequence $\{f_n\}$ of elements of $\mathcal{F}$ contains a subsequence $\{f_{n_k}\}$ that converges uniformly on compact subsets of Ω . That is, $\mathcal{F}$ is **precompact**.

We can now state some important finiteness theorems.

Theorem

20.3

Let X be a compact Riemann surface, and $\pi : E \rightarrow X$ be a holomorphic vector bundle on X . Then, the space $H^0(X, E)$ of global holomorphic of E on X :

$$H^0(X, E) := \{s : X \rightarrow E \mid \pi \circ s = \text{id}_X, s \text{ holomorphic}\}$$

is a finite dimensional $\mathbb{C}$ -vector space.

Proof. Choose finitely many coordinate neighborhood $\{(U_i, z_i)\}_{1 \leq i \leq N}$ on X such that

- (a) $z_i : U_i \rightarrow \Delta := \{z \in \mathbb{C} \mid |z| < 1\}$ is a biholomorphism.
- (b) If $V_i := z_i^{-1}(D(0, \frac{1}{2}))$, then $\bigcup_{i=1}^N V_i = X$
- (c) There are neighborhoods W_i of $\overline{U_i}$, and trivializations of E :

$$\varphi_i : \pi^{-1}(W_i) \rightarrow W_i \times \mathbb{C}^n$$

with corresponding transition functions $g_{ij} : W_i \cap W_j \rightarrow \text{GL}_n \mathbb{C}$.

Any section $s \in H^0(X, E)$ is represented by a family $\{f_i\}_{1 \leq i \leq n}$ of holomorphic functions $f_i : W_i \rightarrow \mathbb{C}^n$ such that $f_i(x) = g_{ij}(x)f_j(x)$, for all $x \in W_i \cap W_j$. Let

$$\|s\|_U := \max_i \sup_{x \in U_i} |f_i(x)|, \quad \|s\|_V := \max_i \sup_{x \in V_i} |f_i(x)|$$

Remark 20.4. One can quickly conclude the proof as follows: the space $\mathcal{X} := (H^0(X, E), \|\cdot\|_U)$ is a Banach space, where completeness of the norm follows from Weierstrass's theorem. Then, by Montel's theorem, the unit ball

$$B_{\mathcal{X}}(0, 1) := \{s \in H^0(X, E) \mid \|s\|_U \leq 1\}$$

is sequentially compact, hence compact. But a Banach space with a compact unit ball is finite dimensional.

That finishes the proof, but there's a more explicit proof due to Siegel, which generalizes to higher dimensions as well:

Claim 20.5. There exists a constant $C > 0$ such that $\|s\|_U \leq C \|s\|_V$, for all $s \in H^0(X, E)$.

Proof. Let $x_0 \in \overline{U_i}$ be such that $|f_i(x_0)| = \|s\|_U$. Choose j so that $x_0 \in V_j$. Then we have

$$|f_i(x)| = |g_{ij}f_j(x_0)| \leq C |f_j(x_0)| \leq C \|s\|_V, \quad C = \max_{1 \leq i, j \leq N} \left\{ \sup_{x \in U_i \cap U_j} \|g_{ij}(x)\| \right\},$$

which proves the claim □

Claim 20.6 (Schwartz' Lemma). Let $s \in H^0(X, E)$, and suppose that $\text{ord}_{a_i}(s) \geq k$, where $k \geq 0$ is a fixed integer, and $a_i \in U_i$ is such that $z_i(a_i) = 0$, for all $i \in \{1, \dots, n\}$. Then, $\|s\|_V \leq 2^{-k} \|s\|_U$.

Proof. In fact f_i/z_i^k is holomorphic on U_i for all i , hence by the maximum principle

$$\sup_{V_i} \left| \frac{f_i}{z_i^k} \right| \leq \sup_{\partial U_i} \left| \frac{f_i}{z_i^k} \right| = \sup_{\partial U_i} |f_i| \leq \|s\|_U.$$

Hence if $x \in V_i$, then

$$|f_i(x)| \leq \sup_{z_i \in V_i} \left(|z_i|^k \left| \frac{f_i}{z_i^k} \right| \right) \leq 2^{-k} \cdot \|s\|_U,$$

proving the claim. $\square$

We deduce that if $s \in H^0(X, E)$ satisfies $\text{ord}_{a_i}(s) \geq k$ for $i \in \{1, \dots, n\}$, the

$$\|s\|_U \leq C \|s\|_V \leq \frac{C}{2^k} \|s\|_U.$$

Thus, if $2^k > C$, we deduce that $s = 0$. Now, let $\mathcal{O}_{a_i}$ denote the ring of germs of holomorphic functions at a_i , and let $\mathfrak{m}_i^k$ be the principal ideal in $\mathcal{O}_{a_i}$ generated by z_i^k . Then $\mathcal{O}_{a_i}/\mathfrak{m}_i^k$ is a $\mathbb{C}$ -vector space of dimension k . If $2^k > C$, the map

$$\begin{aligned} H^0(X, E) &\xrightarrow{\varphi} \bigoplus_{i=1}^N \mathbb{C}^n \otimes (\mathcal{O}_{a_i}/\mathfrak{m}_i^k) \\ s &\mapsto \bigoplus_i (f_i \bmod z_i^k) \end{aligned}$$

is injective, since

$$\ker \varphi = \{s \mid \text{ord}_{a_i}(s) \geq k, \forall i \in \{1, \dots, N\}\}.$$

This proves the theorem. $\square$

Theorem

20.7

Let X be a compact Riemann surface, and $\pi : E \rightarrow X$ be a holomorphic vector bundle on X . Then, $H^1(X, E)$ is a finite dimensional $\mathbb{C}$ -vector space.

Proof. Suppose that (U, φ_U) is a local trivialization of E , so that $\varphi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{C}^n$ is biholomorphic. Suppose $V \subset U$ is a relatively compact open subset of U (meaning $\bar{V} \subset U$ is compact), and let $E_b(V)$ denote the space of bounded holomorphic sections of E on V , i.e. all $s \in \Gamma(V, E)$ with $\varphi_U(s(x)) = (x, f(x))$ where $x \in V$, $f(x) \in \mathbb{C}^n$, and $\sup_{x \in V} |f(x)| < \infty$. Let $\|s\|_V := \sup_{x \in V} |f(x)|$ for $s \in E_b(V)$. This norm makes $E_b(V)$ into a Banach space.

Note. A different local trivialization $\varphi_U^1 : \pi^{-1}(U) \rightarrow U \times \mathbb{C}^n$ gives rise to an equivalent norm on $E_b(V)$, so our choice doesn't matter.

Claim 20.8. If $U \simeq \Omega \subset \mathbb{C}$, with $\simeq$ denoting biholomorphism, then $H^1(U, E) = 0$.

Proof. Mittag-Leffler's theorem (Corollary 14.10) asserts that $H^1(\Omega, \mathcal{O}) = 0$, where $\mathcal{O}$ is the sheaf of holomorphic functions. Now, if $U \simeq \Omega \subset \mathbb{C}$ and $\varphi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{C}^n$ is a trivialization, then

$$H^1(U, E) = H^1\left(U, \bigoplus^n \mathcal{O}\right) \simeq \bigoplus^n H^1(\Omega, \mathcal{O}) = 0$$

since $E|_U \simeq \bigoplus^n \mathcal{O}_U$. $\square$

Now, the claim implies that we can use Leray's theorem to compute $H^1(X, E)$ via a suitable open cover of X : choose a family $\{(W_i, z_i)\}_{1 \leq i \leq N}$ of coordinate neighborhoods on X , and trivializations $\varphi_i : \pi^{-1}(W_i) \rightarrow W_i \times \mathbb{C}^n$ of E on W_i so that

- (i) $z_i : W_i \rightarrow D(0, 2)$ is a biholomorphism, and
- (ii) if $U_i(t) := z_i^{-1}(D(0, t))$, then $\bigcup_{i=1}^N U_i(1/2) = X$.

For each $t \in [1/2, 2]$, let $\mathcal{U}(t)$ denote the open cover $\{U_i(t)\}_{1 \leq i \leq N}$ of X . For every $x \in W_j$ and $v \in E_x := \pi^{-1}(x)$, define $|\varphi_j(v)| := |w|$ where $\varphi_j(v) = (x, w)$ with $w \in \mathbb{C}^n$. We now define the bounded cocycles/coboundaries as follows:

$$Z_b^1(t) := \left\{ \xi \in Z^1(\mathcal{U}(t), E) \mid f_{ij} \in E_b(U_i(t) \cap U_j(t)), \forall i, j \right\}.$$

$$C_b^0(t) := \left\{ \eta \in C^0(\mathcal{U}(t), E) \mid \text{if } \eta = (c_i), \text{ then } c_i \in E_b(U_i(t)), \forall i \right\}$$

Introduce norms $\|\cdot\|_t$ on $Z_b^1(t)$ and $C_b^0(t)$ by

$$\|\xi\|_t := \max_{i,j} \sup_{x \in U_i(t) \cap U_j(t)} |\varphi_i(f_{ij}(x))|, \quad \text{for } \xi = (f_{ij}) \in Z_b^1(t),$$

$$\|\eta\|_t := \max_i \sup_{x \in U_i(t)} |\varphi_i(c_i(x))|, \quad \text{for } \eta = (c_i) \in C_b^0(t).$$

With these normals, $Z_b^1(t)$ and $C_b^0(t)$ are Banach spaces. Let $\frac{1}{2} \leq \rho \leq t < 1$.

Claim 20.9. *If $\eta \in C^0(\mathcal{U}(t), E)$ and $\delta_\eta \in Z_b^1(t)$, then $\eta \in C_b^0(t)$. Moreover, there exists a constant $C > 0$, depending only on $\{W_i, z_i, \varphi_i\}$ such that*

$$\|\eta\|_t \leq \|\delta_\eta\|_t + C \|\eta\|_\rho$$

Proof. Suppose $\eta = (c_i)$ and $x_0 \in U_i(t)$. Choose a j so that $x_0 \in U_j(\rho)$. Then, we have

$$\varphi_i(c_j(x_0)) = \varphi_i \circ \varphi_j^{-1}(\varphi_j(c_j(x_0))) \implies |\varphi_i(c_j(x_0))| \leq C |\varphi_j(c_j(x_0))| \leq C \|\eta\|_\rho$$

where

$$C := \sup_{\substack{i,j \\ x \in U_i(1) \cap U_j(1)}} \|\varphi_i \circ \varphi_j^{-1}(x)\|.$$

Now,

$$c_i(x_0) = (c_i - c_j)(x_0) + c_j(x_0) \implies |\varphi_i(c_i(x_0))| \leq \|\delta_\eta\|_t + C \|\eta\|_\rho,$$

proving the claim. □

Claim 20.10. *Define the bounded cohomology group $H_b^1(t) := Z_b^1(t) / \delta C_b^0(t)$. Then, the natural map*

$$H_b^1(t) \rightarrow \frac{Z^1(\mathcal{U}(t), E)}{\delta C^0(\mathcal{U}(t), E)} = H^1(\mathcal{U}(t), E) \simeq H^1(X, E)$$

is an isomorphism for $t \in [1/2, 1]$.

Proof. [Claim 20.9](#) shows that the map is injective. For surjectivity: we may use Leray's theorem, which implies that the map $H^1(\mathcal{U}(2), E) \xrightarrow{i} H^1(\mathcal{U}(t), E)$ is an isomorphism for all $t \in [1/2, 1]$. Notice that i factors through $H_b^1(t)$, since $D(0, 1)$ is relatively compact in $D(0, 2)$, implying surjectivity. □

Note also that the restriction map $H_b^1(1) \rightarrow H_b^1(t)$ is an isomorphism for all $t \in [1/2, 1]$, so in particular the map $\Phi : Z_b^1(1) \rightarrow H_b^1(t)$ induced by the restriction $Z_b^1(1) \rightarrow Z_b^1(t)$ is surjective.

Observe that $Z_b(1)$ is a Banach space, and we want to know that $H_b^1(t) = Z_b^1(t)/\delta C_b^0(t)$ is a Banach space.

Claim 20.11. $(H_b^1(t), \|\cdot\|)$ is a Banach space.

Proof. Next time.

□

□