

MATH730: Algebraic Topology

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1. Motivation and Review

Motivation

1.1

Distinguish topological objects, and ultimately classify them.

Let's see some examples where the power of algebraic topology helps us prove important topological properties.

Theorem

1.2

The unit ball $\overline{\mathbb{D}}^{n+1}$ is not homeomorphic to S^n

Proof. The n^{th} homology group H_n distinguishes them. Namely, $H_n(\overline{\mathbb{D}}^{n+1}) = \{0\}$, while $H_n(S^n) = \mathbb{Z}$. $\square$

Theorem

1.3

There does not exist a continuous map $f : \overline{\mathbb{D}}^{n+1} \rightarrow S^n$ such that $f(x) = x$ for every $x \in S^n$ (note that S^n is the boundary of $\overline{\mathbb{D}}^{n+1}$).

Proof. Suppose such an f exists. Consider the inclusion map $i : S^n \rightarrow \overline{\mathbb{D}}^{n+1}$. Then, the map $f \circ i : S^n \rightarrow S^n$ is the identity map. Moreover, i and f induce maps on the homology groups

$$H_n(S^n) \xrightarrow{H_n(i)} H_n(\overline{\mathbb{D}}^{n+1}) \xrightarrow{H_n(f)} H_n(S^n).$$

By functoriality,

$$H_n(f) \circ H_n(i) = H_n(f \circ i) = H_n(\text{id}) = \text{id}.$$

Hence, we have

$$\mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z}$$

induces the identity map $\mathbb{Z} \rightarrow \mathbb{Z}$, which is impossible. $\square$

Brauer's Fixed Point Theorem

1.4

Every continuous function $f : \overline{\mathbb{D}}^n \rightarrow \overline{\mathbb{D}}^n$ has a point x such that $f(x) = x$.

Proof. Suppose f does not have a fixed point. Then, for every $x \in \overline{\mathbb{D}}^n$, we may consider the unique straight line from $f(x)$ to x . Let $g(x)$ be the point where the line $f(x) \rightarrow x$ hits S^n . Observe that $g(x) = x$ for every $x \in S^n$, and g is continuous. Thus, we have a contradiction by [Theorem 1.3](#). $\square$

1.1. Topology Review

Definition

1.5

In $\mathbb{R}^n$, we have the usual metric $d(x, y)$. We denote $B(x, r)$ to be the open ball around x with radius r .

Definition

1.6

A topological space X is called **compact** if every open cover has a finite subcover.

Theorem

1.7

A subset $X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded.

Proof. Suppose X is compact. The set $\{B(0, n)\}_{n=1}^{\infty}$ of open balls is necessarily an open cover of X , hence has a finite subcover, thus X is bounded. Moreover, if $\mathbb{R}^n - X$ is not open, let $p \in \mathbb{R}^n \setminus X$ be such that $B(p, \varepsilon) \cap X \neq \emptyset$. Then, take the cover of X by

$$\left\{ B\left(x, \frac{d(p, x)}{2}\right) \right\}_{x \in X}.$$

One sees that this does not have a finite subcover, a contradiction.

For the other direction, note that closed+bounded implies sequential compactness. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X .

Claim 1.8. *There exists $\delta > 0$ such that for every $x \in X$, $B(x, \delta) \subset U_\alpha$ for some $\alpha \in I$.*

Proof. Contrarily, if this were not the case, there exists a sequence $\{x_i\} \subset X$ such that $B(x_i, 1/n) \not\subset U_\alpha$ for all $\alpha \in I$. Then, $\{x_i\}$ has a subsequence converging to some p . This p must be contained in some U in the open cover, and it follows that there exists a δ and N with $B(x_N, \delta) \subset U$. $\square$

Since X is bounded in $\mathbb{R}^n$, for every $\delta > 0$ the number of disjoint balls of radius δ around points in X is bounded. Let m be this upper bound. Take a point $p_1 \in X$. The ball $B(p_1, \delta)$ is contained in some cover element U_1 . If U_1 covers X , we're done, but otherwise repeat with some p_2 and U_2 . Continue to repeat until we have p_{m+1} and U_{m+1} . We claim that $U_1, \dots, U_{m+1}$ must cover X : indeed, if it doesn't, then the balls $\{B(p_i, \delta/2)\}_{i=1}^{m+1}$ must all be disjoint, and this is a contradiction. $\square$

Definition

1.9

A topological space X is called **connected** if $X = U_1 \sqcup U_2$ with U_1, U_2 open implies $U_1 = \emptyset$ or $U_2 = \emptyset$.

Definition

1.10

X is called **path connected** if for every $p, q \in X$, there exists a continuous map $\delta : [0, 1] \rightarrow X$ such that $\delta(0) = p$ and $\delta(1) = q$.

Proposition**1.11**Path connected $\implies$ connected.**Definition****1.12**

A topological space X is called **Hausdorff** if for every distinct $p, q \in X$, there exists $U \ni p$ and $V \ni q$ open such that $U \cap V = \emptyset$.

Theorem**1.13**

Let $f : X \rightarrow Y$ be bijective and continuous, with X compact, Y Hausdorff. Then, f is a homeomorphism.

Proof. It suffices to show f is an closed map. Let $C \subset X$ be closed. Since X is compact, one can see that C is compact (theorem). Therefore, $f(C)$ is compact (theorem), hence closed (theorem). $\square$

Proposition**1.14**

If X is Hausdorff and $A \subset X$ is compact, then A is closed.

Exercise 1. Give an example of a space X such that $A \subset X$ is compact but not closed. This space cannot be Hausdorff.

Definition**1.15**

Let X be a topological space, and $\sim$ be an equivalence relation on X . The **quotient-topology** is defined as follows: let $\pi : X \rightarrow X/\sim$ be the projection map. Then, $U \subset X/\sim$ is open whenever $\pi^{-1}(U)$ is open.

Proposition**1.16**

A map $f : (X/\sim) \rightarrow Y$ is continuous if and only if $f \circ \pi$ is continuous.

Example 1.17. Let $\mathbb{RP}^n$ be the space of lines through the origin in $\mathbb{R}^{n+1}$: this is the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence $x \sim y \iff$ there exists $\lambda \in \mathbb{R}$ with $\lambda x = y$. This is homeomorphic to the quotient space of S^n by the relation $x \sim -x$.

Definition**1.18**

Let $(X_\alpha, \mathcal{U}_\alpha)_{\alpha \in I}$ be topological spaces. The **product topology** on $\prod_I X_\alpha$ has a basis of sets of the form

$$\prod_{\alpha \in I} U_\alpha, \quad U_\alpha \in \mathcal{U}_\alpha,$$

and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in I$.

Tychonoff's Theorem**1.19**

The product of compact spaces is compact.

2. CW Complexes

Definition

2.1

A **CW complex** is a topological space X , together with a sequence of subspaces $X_0 \subset X_1 \subset \cdots$ such that

1. $X = \bigcup X_n$
2. $U \subset X$ is open if and only if $U \cap X_n$ is open for all $n \geq 0$
3. X_0 is a discrete set of points
4. For every $n \geq 1$, there exists an indexing set $I(n)$, disks $\{D_\alpha^n\}_{\alpha \in I(n)}$, and continuous maps $\varphi_\alpha^n : \partial D_\alpha^n \rightarrow X_{n-1}$ such that

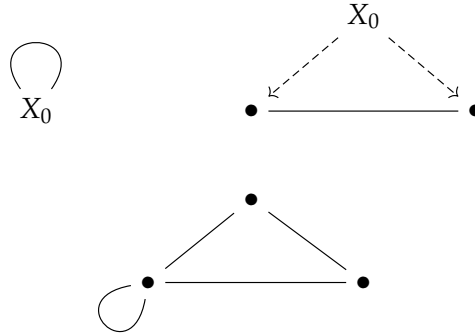
$$X_n = \bigsqcup_{\alpha \in I(n)} \overline{D}_\alpha^n \sqcup X_{n-1} / \sim$$

where $x \sim \varphi_\alpha^n(x)$ for all $\alpha \in I(n)$, $x \in \partial D_\alpha^n$.

We call X_n the n^{th} **skeleton** of X , and φ_α^n is called an **attaching map**. For a CW complex X , the sets $\{X_n\}$ and $\{\varphi_\alpha^n\}$ are called a **CW structure** on X . The quotient maps $\Phi_\alpha^n : \overline{D}_\alpha^n \rightarrow X_n$ are called the **characteristic maps**. If $X_{n-1} \subsetneq X_n$, but $X_i = X_j$ for all $i, j \geq n$, we say that X is **n -dimensional**.

Example 2.2. A discrete set of points is a 0-dimensional CW complex.

Example 2.3. A graph is a 1-dimensional CW complex.



Example 2.4. $\mathbb{R}^n$ is a CW complex. For example, $\mathbb{R}^2$ has the 0-cell $\mathbb{Z}^2$, the 1-cell a $\mathbb{Z}^2$ -grid, and the 2-cell induced by attaching each disk to the square of the grid.

Example 2.5. A torus is a CW complex.

Example 2.6. S^n is a CW complex. Namely, X_0 is a single point, $X_1 = \cdots = X_{n-1} = X_0$, and the (only possible) attaching map $\varphi_1^n : D_1^n \rightarrow X_{n-1}$.

Example 2.7. $\mathbb{R}P^n = S^n / (x \sim -x)$ is a CW complex. When $n = 0$, $\mathbb{R}P^n$ is just a point, hence as a CW structure. Suppose $\mathbb{R}P^{n-1}$ has a CW structure. Let $\overline{D}_\alpha^n$ be the upper hemisphere. We obtain the CW structure on $\mathbb{R}P^n$ by attaching one n -cell to $\mathbb{R}P^{n-1}$ via the attaching map

$$\varphi_\alpha^n(x_1, \dots, x_n, 0) = [(x_1, \dots, x_n)] \subset \mathbb{R}P^{n-1}.$$

Exercise 2. Find a CW structure on $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / (x \sim \lambda x, \lambda \in \mathbb{C}^*)$

Lemma**2.8**

Let X be a CW-complex. Then, $U \subset X$ is open $\iff$ for every n and $\alpha \in I(n)$, $(\Phi_\alpha^n)^{-1}(U)$ is open.

Proof. Recall that U is open in X if and only if $U \cap X^n$ is open for all n , if and only if for the projection map

$$P^n : X^n \sqcup \bigsqcup_{\alpha \in I(n)} D_\alpha^n \rightarrow X^n,$$

we have

$$(P^n)^{-1}(U \cap X^n)$$

is open. This is the case if and only if for each $\alpha \in I(n)$,

$$(\Phi_\alpha^n)^{-1}(U \cap X)$$

and $U \cap X^{n-1}$ are open. So, $(\implies)$ is verified, and $(\impliedby)$ follows by induction on n . $\square$

Lemma**2.9**

Let $K \subset X$ be compact. Then, K intersects only finitely many cells in X .

Proof. Suppose not. Then, we can choose an infinite set $S \subset K$ such that no two points in S belong to the same cell. We claim S is closed. Indeed, note that $S \cap X^0$ is closed since X^0 is discrete. By induction, suppose $S \cap X^{n-1}$ is closed. Then, $S \cap X^n$ is closed $\iff (\Phi_\alpha^n)^{-1}(S)$ is closed, and by induction $(\Phi_\alpha^n)^{-1}(S) \cap \partial D_\alpha^n$ is closed, so indeed $(\Phi_\alpha^n)^{-1}(S)$ is closed as the union of closed sets. Thus, $S \cap X^n$ is closed for every n , hence S is closed.

In fact, we've shown that every subset of S is closed, so S is discrete. Moreover, S is compact as a closed subset of a compact set. Thus, S is compact and discrete, hence finite. $\square$

Definition**2.10**

Let $A \subset X$ be a set, and fix $\varepsilon := \{\varepsilon_\alpha^n \in \mathbb{R}^+\}$ for every n and $\alpha \in I(n)$. Then, an ε -neighborhood of A is the open set $N_\varepsilon(A) \subset X$ defined as follows: let

$$N_\varepsilon^0(A) := A \cap X^0$$

and then for every $\alpha \in I(n)$, set

$$(\Phi_\alpha^n)^{-1}(N_\varepsilon^n(A)) := [B(A \cap \text{Int}(D_\alpha^n), \varepsilon_\alpha^n) \cap \text{Int}(D_\alpha^n)] \cup \left[(1 - \varepsilon_\alpha^n, 1](\Phi_\alpha^n)^{-1}(N_\varepsilon^{n-1}(A)) \right],$$

and

$$N_\varepsilon(A) = \bigcup N_\varepsilon^n(A).$$

Proposition**2.11**

$$N_\varepsilon(A) \cap X^n = N_\varepsilon^n(A)$$

Theorem**2.12**

Let X be a CW complex. Then, X is Hausdorff.

Proof. Let $p \neq q \in X$, and suppose that $p, q \in X^n$ but $q \notin X^{n-1}$. So, q is in some disk e_α^n . If $p \in X^{n-1}$, we may choose ε sufficiently small so that $N_\varepsilon(X^{n-1})$ and $N_\varepsilon(q)$ are disjoint. If $p \notin X^{n-1}$, then p, q are distinct points in e_α^n which is Hausdorff, hence separable in ε -neighborhoods. These ε -neighborhoods lift to the global structure of X so that $N_\varepsilon(p) \cap N_\varepsilon(q) = \emptyset$. $\square$

Definition**2.13**

Let X, Y be topological spaces. Two (continuous) maps $f, g : X \rightarrow Y$ are **homotopic** if there exists $H : X \times [0, 1] \rightarrow Y$ continuous such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for every $x \in X$.

For a subset $A \subset X$, two maps $f, g : X \rightarrow Y$ are called **homotopic relative to A** if $f|_A = g|_A$, and there exists a homotopy $H : X \times [0, 1] \rightarrow Y$ between f, g such that $H(x, t) = f(x) = g(x)$ for all $x \in A, t \in [0, 1]$.

Homotopy is denoted $f \sim g$ or $f \stackrel{H}{\sim} g$.

Definition**2.14**

Two maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are called **homotopy equivalent** if $g \circ f \sim \text{id}_X$ and $f \circ g \sim \text{id}_Y$. In this case, X and Y are said to be **homotopically equivalent**.

A subset $A \subset X$ is called a **retract** of X if there exists $r : X \rightarrow A$ continuous such that $r(x) = x$ for every $x \in A$. Moreover, A is called a **deformation retract** if $r \sim \text{id}_X$ relative to A . X is called **contractible** if $\text{id}_X \sim \text{constant}$.

Example 2.15. Show that $\mathbb{R}^{n+1} \setminus \{0\}$ deformation retracts to $S^n \subset \mathbb{R}^{n+1}$.

Example 2.16. Show that $i : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$ and $\pi : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$ are homotopy equivalent.

Example 2.17. Let X be a topological space. Define $C(X) = X \times [0, 1] / [(x, 1) \sim (y, 1)]$. Show that $C(X)$ is contractible.

Definition**2.18**

X is called **locally contractible** if for every $p \in X$ and $U \ni p$ open, there exists $V \subset U$ with $p \in V$ such that p is contractible.

Proposition**2.19**

Every CW complex is locally contractible.

Proof. Exercise. Inductively push the additional part of the neighborhood to the boundary. $\square$

Corollary**2.20**

If X is a CW complex, then X is connected $\iff$ it's path connected.

Proof. Exercise. Local contractibility allows you to understand path connectivity because of the path induced by a retract. $\square$

3. Fundamental Group

Definition

3.1

Let X be a topological space $p \in X$. The **fundamental group** of X at p , is

$$\pi_1(X, p) := \{ \gamma : [0, 1] \rightarrow X \mid \gamma(0) = \gamma(1) \} / \left[\gamma \stackrel{\text{rel}(\{0,1\})}{\sim} \mu \right].$$

Equivalently, let q be a fixed point in S^1 , and we have

$$\pi_1(X, p) = \{ \gamma : S^1 \rightarrow X \} / \left[\gamma \stackrel{\text{rel}(q)}{\sim} \mu \right]$$

This definition is in some ways better, since changing S^1 to S^n gives a more generalized definition of the **n -th fundamental group** $\pi_n(X, p)$.

Definition

3.2

A connected space X is called **simply connected** if there exists $p \in X$ such that $\pi_1(X, p) = \{[p]\}$.

Proposition

3.3

If X is contractible, then

$$\pi_1(X, p) = \{[p]\}$$

for the p such that X deformation retracts to p .

Proof. Let $\gamma \in \pi_1(X, p)$. Let H be a homotopy between id_X and p . Take

$$\tilde{H}(s, t) := H(\gamma(s), t).$$

Then, $\gamma \stackrel{\tilde{H}}{\sim} p$. $\square$

Definition

3.4

For $\gamma, \mu \in \pi_1(X, p)$, define

$$\gamma \star \mu(t) = \begin{cases} \mu(2t) & 0 \leq t \leq 1/2 \\ \gamma(2t - 1) & 1/2 \leq t \leq 1 \end{cases}.$$

Then, $\star$ induces a well-defined group structure on $\pi_1(X, p)$.

Indeed, in order to show that this concatenation of curves defines a group structure on $\pi_1(X, p)$, one must prove the following claim:

Claim 3.5. *If $[\gamma] = [\gamma']$ and $[\mu] = [\mu']$, then $[\gamma \star \mu] = [\gamma' \star \mu']$. Thus, $[\gamma] \star [\mu]$ is a well-defined binary operation on $\pi_1(X, p)$. Moreover, $\pi_1(X, p)$ is a group, with identity the constant map $[p]$, and*

$$[\gamma(t)]^{-1} = [\gamma(1-t)].$$

Exercise 3. Show that if $\varphi : [0, 1] \rightarrow [0, 1]$ is a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$, then $\gamma \circ \varphi \sim \gamma$.

Theorem

3.6

Let X be path connected, $p, q \in X$. Then, $\pi_1(X, p) \cong \pi_1(X, q)$.

Proof. Since X is path connected, we have a continuous curve $\mu : [0, 1] \rightarrow X$ with $\mu(0) = p$ and $\mu(1) = q$.

Claim 3.7. Let Ω_p denote the space of curves starting and ending at p . Then, the map

$$T_\mu : \Omega_p \rightarrow \Omega_q$$

defined by $T_\mu := \mu \star \gamma \star \mu^{-1}$ induces an isomorphism $\pi_1(X, p) \xrightarrow{\sim} \pi_1(X, q)$.

Indeed, suppose $[\gamma] = [\gamma']$. Then, $[T_\mu(\gamma)] = [T_\mu(\gamma')]$. Moreover,

$$\mu \star (\gamma_1 \star \gamma_2) \star \mu^{-1} = \mu \star \gamma_1 \star \mu^{-1} \star \mu \star \gamma_2 \star \mu^{-1},$$

hence T_μ is a homomorphism. Moreover, the map $\psi = \mu(1-t)$ induces a clear inverse homomorphism $T_\psi : \pi_1(X, q) \rightarrow \pi_1(X, p)$, so it follows that T_μ is an isomorphism. $\square$

Proposition

3.8

Let (X, p) and (Y, q) be (pointed) topological spaces, and $f : (X, p) \rightarrow (Y, q)$ be continuous (i.e. a continuous map $f : X \rightarrow Y$ with $f(p) = q$). Then, the map $f_* := \pi_1(f) : \pi_1(X, p) \rightarrow \pi_1(Y, q)$ defined by

$$f_*([\gamma]) = [f \circ \gamma]$$

is a well-defined group homomorphism. Moreover, if

$$(X, p) \xrightarrow{f} (Y, q) \xrightarrow{g} (Z, r)$$

is a chain of continuous maps, then $(g \circ f)_* = g_* \circ f_*$.

Proof. Let $[\gamma] = [\gamma']$. Then, $f \circ \gamma \sim f \circ \gamma'$ on $f \circ H$, relative to $\{0, 1\}$, so this is well-defined, and

$$(f \circ \gamma) \star (f \circ \mu) = f(\gamma \star \mu)$$

at the curves level, which induces the homomorphism $(g \circ f)_* = g_* \circ f_*$ at the level of fundamental groups. $\square$

Functoriality of the Fundamental Group

3.9

The map π_1 on spaces and the continuous maps between them is a functor from the category of pointed spaces to the category of groups. That is,

$$\pi_1 : \text{Top}_\bullet \rightarrow \text{Grp}$$

is a covariant functor.

Definition

3.10

X is called **strongly contractable** if there exists $p \in X$ such that X deformation retracts to p .

Proposition

3.11

If X is strongly contractable, then $\pi_1(X, q) = \{e\}$ for all $q \in X$.

Proof. When $q = p$, we know $\text{id}_X \stackrel{\text{rel } p}{\sim} (x \mapsto p)$, and it follows that $\pi_1(X, p)$ is trivial. Then, the theorem follows for arbitrary q by [Theorem 3.6](#), since strongly contractable implies path connected. $\square$

Recall

X is called contractible if $X \sim \{q\}$, i.e. $\text{id}_X \sim (x \mapsto q)$.

Theorem

3.12

If X is contractible, then $\pi_1(X, q) = \{e\}$ for all $q \in X$.

4. The Fundamental Group of S^1 **Theorem**

4.1

Let $\gamma : [0, 1] \rightarrow S^1$ be continuous with $\gamma(0) = 1$. Then, there exists a unique $\theta : [0, 1] \rightarrow \mathbb{R}$ such that $\theta(0) = 0$, and

$$e^{i\theta(t)} = \gamma(t).$$

Moreover, we can do this continuously along homotopies. That is, if $H : [0, 1] \times [0, 1] \rightarrow S^1$ has $H(0, s) = 1$, there there exists a unique $\theta : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ such that $\theta(0, s) = 0$ for all s , and $e^{i\theta} = H$ for all t, s .

Corollary

4.2

Let $\gamma_1, \gamma_2 : [0, 1] \rightarrow S^1$ be such that $\gamma_i(0) = \gamma_i(1) = 1$, $i = 1, 2$. Let θ_1, θ_2 be their lifts as in [Theorem 4.1](#). Then,

$$\gamma_1 \stackrel{\text{rel}\{0,1\}}{\sim} \gamma_2 \iff \theta_1(1) = \theta_2(1)$$

Proof. ($\implies$) Suppose $\gamma_1 \sim \gamma_2$ relative to $\{0, 1\}$, via a homotopy H . Let $\theta : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be the lift of H as in [Theorem 4.1](#). Then, $\theta_1(x) = \theta(x, 0)$ and $\theta_2(x) = \theta(x, 1)$. Consider the path $s \mapsto \theta(1, s)$. Then, since $H(1, s) = 1$, and $e^{i\theta} = H$, we have $\theta(1, s) \in 2\pi\mathbb{Z}$. So, as $\theta(1, s)$ is continuous with a discrete image, it must be constant, hence $\theta_1(1) = \theta_2(1)$.

($\impliedby$) Take $\theta(t, s) = (1 - s)\theta_1(t) + s\theta_2(t)$. Then, $\theta(t, s)$ is a homotopy between θ_1 and θ_2 , with $\theta(1, s) = \theta_1(1)$, and $\theta(0, s) = 0$. Now, we may take $H(t, s) := e^{i\theta(t, s)}$, and H is the desired homotopy from γ_1 to γ_2 relative to $\{0, 1\}$. Note that relative to 1 comes from the fact $\theta_1(1) = \theta_2(1)$. $\square$

Corollary

4.3

For any $p \in S^1$,

$$\pi_1(S^1, p) \cong \mathbb{Z},$$

with $t \mapsto e^{2\pi it} : [0, 1] \rightarrow S^1$ being the generator.

Proof. Clearly, $(e^{2\pi it})^k \stackrel{\text{rel}\{0,1\}}{\sim} e^{2\pi ikt}$, where the LHS denotes concatenation k times. On the other hand, each element in $\pi_1(S^1, 1)$ is in bijective correspondence with $t \mapsto e^{2\pi ikt}$ for some $k \in \mathbb{Z}$. Indeed, if θ is the lift of an element in $\pi_1(S^1, 1)$, then $\theta_1(1) \in 2\pi\mathbb{Z}$, and the bijective correspondence is induced by $\gamma \leftrightarrow \frac{\theta_1(1)}{2\pi}$. $\square$

Exercise 4. Let $\{(X_i, p_i)\}_{i=1}^N$ be a set of pointed spaces. Show that

$$\pi_1 \left(\prod_{k=1}^N X_i, (p_1, \dots, p_n) \right) \cong \prod_{k=1}^N \pi_1(X_i, p_i).$$

Exercise 5. Show that a continuous map $f : \overline{D}^2 \rightarrow S^1$ satisfying $f(x) = x$ for all $x \in S^1$ does not exist.

Proposition

4.4

For all $k \geq 2$, $\pi_1(S^k, p)$ is trivial.

Proof. Let $\gamma : [0, 1] \rightarrow S^k$ be continuous. We claim γ is homotopic, relative to $\{0, 1\}$, to a curve $\tilde{\gamma}$ that misses a point q . The proposition follows from the claim: indeed, we have $\psi : S^k \setminus \{q\} \xrightarrow{\sim} \mathbb{D}^k$, so consider $\mu := \pi \circ \tilde{\gamma}$, which is homotopic to the constant map, and we may push it back to S^k via ψ^{-1} to show $[\gamma] = [\tilde{\gamma}] = [\text{id}]$.

To prove the claim, we approximate γ via smooth maps, and smooth maps always miss points by [Sard's theorem](#). $\square$

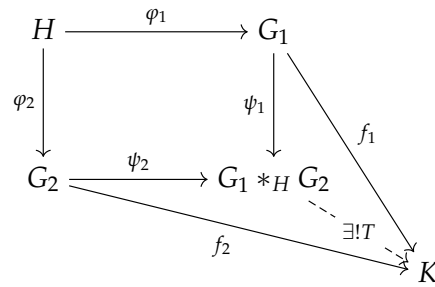
5. Amalgamated Products and Free Products

Definition

5.1

Let H, G_1, G_2 be groups, and let $\varphi_i : H \rightarrow G_i$ be homomorphisms. The **amalgamated product** of G_1 and G_2 along H , denoted $G_1 *_H G_2$, is a group together with homomorphisms $\psi_i : G_i \rightarrow G_1 *_H G_2$ satisfying $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$. Moreover, it has the universal property that for every group K and homomorphisms $f_i : G_i \rightarrow K$ with $f_1 \circ \varphi_1 = f_2 \circ \varphi_2$, there exists a unique homomorphism $T : G_1 *_H G_2 \rightarrow K$ such that $T \circ \psi_i = f_i$. That is, the following diagram

commutes:



A good way to think about this is gluing G_1 and G_2 along H .

Example 5.2. Consider a similar definition but with sets $A \xrightarrow{\varphi_1} B$ and $A \xrightarrow{\varphi_2} B_2$. If we have $f_1 : B_1 \rightarrow K$ and $f_2 : B_2 \rightarrow K$, then one can see $B_1 \sqcup B_2$ with the natural embeddings $i_1 : B_1 \rightarrow B_1 \sqcup B_2$, $i_2 : B_2 \rightarrow B_1 \sqcup B_2$, has the desired unique $\tilde{T} : B_1 \sqcup B_2 \rightarrow K$. However, we do not have the desired commutativity $i_1 \circ \varphi_1 = i_2 \circ \varphi_2$. In order to have this, we need to quotient $B_1 \sqcup B_2$ by the relation $i_1 \varphi_1(x) \sim i_2 \varphi_2(x)$. Then, $(B_1 \sqcup B_2) / \sim$ is the amalgamated product of these sets.

Existence and Uniqueness of the Amalgamated Product

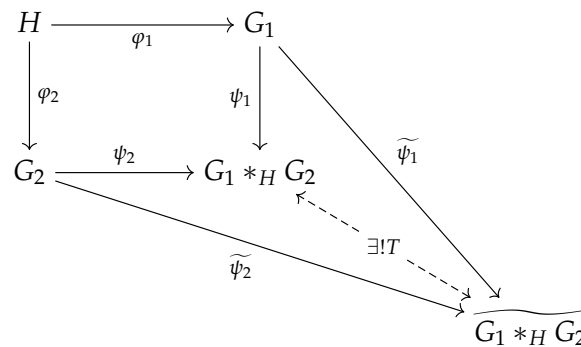
5.3

Let H, G_1, G_2 be groups with homomorphisms $\varphi_i : H \rightarrow G_i$. Then, the amalgamated product $(G_1 *_H G_2, \psi_1, \psi_2)$ exists, and if $(\widetilde{G_1 *_H G_2}, \tilde{\psi}_1, \tilde{\psi}_2)$ is another amalgamated product, then there exists a unique isomorphism

$$T : G_1 *_H G_2 \rightarrow \widetilde{G_1 *_H G_2}$$

such that $T \circ \psi_i = \tilde{\psi}_i$.

Proof.



□

Definition

5.4

Let G_1, G_2 be two groups. The **free product** $G_1 * G_2$ is the group of words in $G_1 \setminus \{e_1\}, G_2 \setminus \{e_2\}$, and a new identity $\{e\}$, where elements from the same group multiply as expected, the product of elements is concatenation, and multiplying an element g from G_1 or G_2 by its inverse in that group yields e .

Theorem

5.5

Let G_1, G_2 be groups. Then,

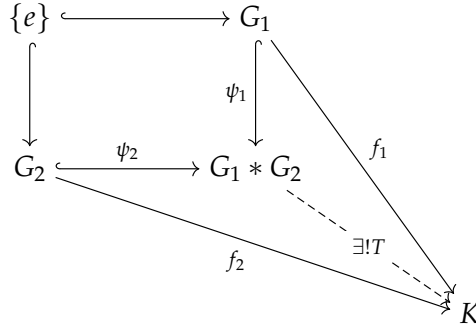
$$G_1 * G_2 \cong G_1 *_{\{e\}} G_2$$

Proof. The claim follows immediately from the following lemma.

Lemma

5.6

The free product $G_1 * G_2$ satisfies the following universal property:



□

Exercise 6. Show that

$$G_1 *_H G_2 \cong (G_1 * G_2) / \langle \varphi_1(h) \varphi_2(h)^{-1} \rangle$$

Remark 5.7. We can similarly define free products $*_{\alpha} G_{\alpha}$ of arbitrarily many groups $\{G_{\alpha}\}_{\alpha \in I}$.

6. Van Kampen's Theorem

Van Kampen's Theorem

6.1

Let (X, p) be a connected pointed space, and suppose

$$X = \bigcup_{\alpha \in I} (U_{\alpha}, p)$$

where $U_{\alpha} \subset X$ is open and $p \in U_{\alpha}$. Suppose that for every $\alpha, \beta \in I$, $U_{\alpha} \cap U_{\beta}$ is path connected. Then, the map

$$\begin{aligned} \Phi : *_{\alpha} \pi_1(U_{\alpha}, p) &\rightarrow \pi_1(X, p) \\ *_{\alpha} [\gamma_{\alpha}] &\mapsto \star_{\alpha} [\gamma_{\alpha}] \end{aligned}$$

is surjective. Moreover, if $\alpha, \beta, \delta \in I$ and $U_{\alpha} \cap U_{\beta} \cap U_{\delta}$ is path-connected, then

$$\pi_1(X, p) \cong *_{\alpha} \pi_1(U_{\alpha}, p) / \langle\langle i_{\alpha, \beta}(\gamma) i_{\beta, \alpha}(\gamma^{-1}) \rangle\rangle,$$

where

$$i_{\alpha, \beta} : \pi_1(U_{\alpha} \cap U_{\beta}, p) \rightarrow \pi_1(U_{\alpha}, p)$$

is the expected embedding, and $\langle\langle - \rangle\rangle$ is the normal closure. In particular, if $(X, p) = U_1 \cup U_2$

with $U_1 \cap U_2$ path connected, then

$$\pi_1(X, p) = \pi_1(U_1, p) *_{\pi_1(U_1 \cap U_2, p)} \pi_1(U_2, p)$$

Example 6.2. Take $X = \bigvee_{\alpha \in I} (S^1, p)$, which is the space defined by attaching I copies of S^1 at one shared point p . For each α , take $U_\alpha = S_\alpha^1 \cup N_\varepsilon(p)$. Then,

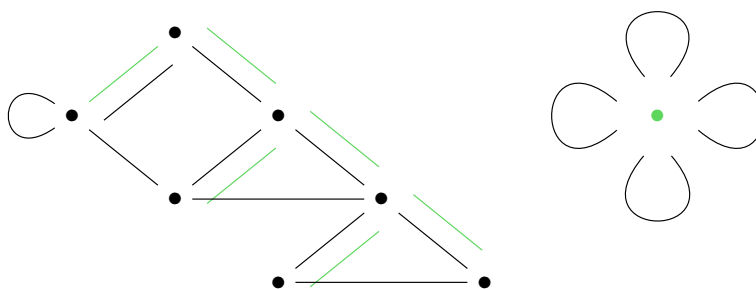
$$U_\alpha \cap U_\beta \cap U_\delta = N_\varepsilon(p)$$

for every $\alpha, \beta, \delta \in I$ is connected (and contractible). Thus, by van Kampen's theorem,

$$\pi_1(X, p) \cong *_\alpha \pi_1(S_\alpha^1, p) \cong *_\alpha \mathbb{Z}$$

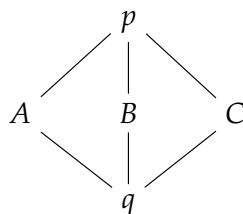
Indeed, the normal subgroup is trivial because $N_\varepsilon(p)$ is contractible.

Example 6.3. Let $G = (V, E)$ be an undirected connected graph, viewed as a CW-complex. Let T be a spanning tree. Then, deforming the tree to a point, we get a bouquet as in the previous example:



Remark 6.4. Recall that we required $U_\alpha \cap U_\beta$ to be connected for surjectivity to hold. If they are not connected, then we get examples of maps which aren't surjective: for instance, taking two arclengths of S^1 , their intersection has two connected components, and $\{e\} * \{e\} \not\rightarrow \mathbb{Z}$.

The triple intersection condition is also necessary. Consider the graph:



Let $U_x = X \setminus \{x\}$. Then, $U_x \cap U_y$ is always contractible, but $U_A \cap U_B \cap U_C$ is disconnected. Notice further that $\pi_1(U_x, p) = \mathbb{Z}$, and $\pi_1(X, p) = \mathbb{Z} * \mathbb{Z}$. Then, van Kampen would say $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} \rightarrow \mathbb{Z} * \mathbb{Z}$, and $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} / \langle e \rangle \cong \mathbb{Z} * \mathbb{Z}$, but this is absurd.

We now prove van Kampen's theorem.

Proof of van Kampen. Let (X, p) be a pointed space, $X = \bigcup_{\alpha \in I} (U_\alpha, p)$, suppose $U_\alpha \cap U_\beta$ is path connected for all α, β , and let Φ be the map

$$\Phi : *_\alpha \pi_1(U_\alpha, p) \rightarrow \pi_1(X).$$

We claim Φ is surjective. Let $\gamma : [0, 1] \rightarrow X$ be a loop at p , and we want to decompose it into loops in (U_α, p) . Since γ is continuous, the set

$$\{\gamma^{-1}(U_\alpha)\}_{\alpha \in I}$$

is an open cover of $[0, 1]$. Thus, there exists $\delta > 0$ called the **Lebesgue number**, such that any open subset of $[0, 1]$ with diameter $< \delta$ is contained in $\gamma^{-1}(U_\alpha)$ for some $\alpha \in I$. Assume $\delta = 1/k$ for $k \in \mathbb{N}$, and partition the interval $[0, 1]$ into $2k$ sub-intervals of length $1/2k$. Thus, each sub-interval $[\frac{i-1}{2k}, \frac{i}{2k}]$ is mapped via γ into U_α . Let $t_i = i/2k$ and $p_i = \gamma(t_i) \in U_{\alpha_i}$. Since $U_{\alpha_i} \cap U_{\alpha_{i+1}}$ is connected, there exists a curve μ_i from p_i to p . Let $\gamma_i = \gamma|_{[t_{i-1}, t_i]}$. Then,

$$\gamma = \gamma_{2k} * \cdots * \gamma_1 \sim \gamma_{2k} \cdots \gamma_3 \mu_2^{-1} \mu_2 \gamma_2 \mu_1^{-1} \mu_1 \gamma_1,$$

proving surjectivity.

Now, in order to prove the desired isomorphism, it suffices to show

$$\ker(\Phi) \cong \langle \langle i_{\alpha\beta}(\gamma) \cdot i_{\beta\alpha}(\gamma^{-1}) \rangle \rangle =: N.$$

Indeed, first observe that

$$\Phi(i_{\alpha\beta}(\gamma) \cdot i_{\beta\alpha}(\gamma^{-1})) = [\gamma][\gamma^{-1}] = e,$$

so clearly $N \subset \ker \Phi$.

For the other direction, we want to show that if $\gamma_1 * \cdots * \gamma_k$ is homotopic to $[e]$ as loops in X , then $\gamma_1 * \cdots * \gamma_k \in N$. This is provable, but the proof was confusing in lecture... I didn't feel like finishing it. $\square$

Exercise 7. Show that $\mathbb{R}^2 \setminus \{p\}$ deformation retracts to S^1 , and conclude that $\mathbb{R}^2 \not\cong \mathbb{R}^k$ for $k > 2$.

Exercise 8. Compute $\pi_1(\mathbb{R}^2 \setminus \{p_1, \dots, p_k\})$ using van Kampen's theorem.

Exercise 9. How do you hang a picture using 1 rope and k nails such that removing any nail will make it fall.

Example 6.5 (The Fundamental Group of $\mathbb{R}^3 \setminus S^1$). We can imagine the sphere of which S^1 is a cross section (S_2), and deformation retract everything outside that sphere down to it (that is, we can ignore everything far outside our circle). We claim that $\bar{B} \setminus S^1$ deformation retracts to $S^2 \cup L$, where B is the closed ball corresponding to our sphere S_2 , and L is a line passing from one end of S^1 to another.

Then, L can be homotoped within the interior of S^2 to a closed loop touching a point of S^2 . That is, our space retracts to $S^2 \vee S^1$, and by van Kampen's theorem

$$\pi_1(\mathbb{R} \setminus \{0\}) = \pi_1(S^2 \vee S^1) = \pi_1(S^1) = \mathbb{Z}.$$

Example 6.6. Now, consider $\mathbb{R}^3$, with multiple separate circles removed. By shrinking $\mathbb{R}^3$ to the balls surrounding each circle, and using the same strategy as the previous example, we see that this space is homotopy equivalent to $S^2 \vee S^1 \vee S^2 \vee S^1 \vee \cdots \vee S^2 \vee S^1$, and so the fundamental group is $\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$.

Example 6.7. Now, consider $\mathbb{R}^3$, with two linked circles removed. We claim this is homotopy equivalent to $T^2 \vee S^2$, so the fundamental group is

$$\pi_1(T^2) * \pi_1(S^2) = \mathbb{Z}^2 * \{e\} = \mathbb{Z}^2.$$

The idea is as follows: we thicken one of our circles so that it becomes a torus, and we surround our torus and circle by a sphere so that the torus touches the boundary of the sphere. Then, the second circle is “jump-roped” around the torus so that it reaches the boundary of the sphere at that same of the intersection point.

Example 6.8. Consider the space $S^n \setminus \{p\}$, for $n \geq 0$. We claim the fundamental group is trivial. Indeed, we may pass S^{n-1} through our point, which splits S^n into an “upper-hemisphere” U and “lower-hemisphere” L . We widen U and L mildly so that they have non-trivial intersection, and $U \cap L \sim S^1$. Then,

$$\pi_1(S^n, p) = \pi_1(U) *_{\pi_1(S^{n-1}, p)} \pi_1(L),$$

which by induction is trivial (since $\pi_1(S^1 \setminus \{p\}) = \{0\}$).

Proposition

6.9

Suppose Y is obtained from a path-connected pointed space (X, x_0) by attaching a 2-cell, i.e.

$$Y = X \sqcup \overline{D^2} / (\varphi(x) \sim x),$$

where $X \in \partial \overline{D^2}$ and $\varphi : \partial D^2 \rightarrow X$. Let $p \in Y \cap X$ be a point such that there exists $\tilde{p} \in D^2$ with $\varphi(\tilde{p}) = p$. Let μ_p be a path in X between x_0 and p . Then,

$$\pi_1(Y, x_0) = \pi_1(X, x_0) / \langle \langle \mu_p^{-1} * \varphi * \mu_p \rangle \rangle.$$

Proof. Let p' be a point in the interior of D obtained by moving radially inward from $\tilde{p}$. Let $\tilde{\varphi}$ be a loop at p' , let φ_1 be a loop in D^2 with $\tilde{\varphi}$ in the interior, and φ_2 be a loop with $\tilde{\varphi}$ in the exterior. Let G be the region within the loop φ_1 , and P the region outside the loop φ_2 . Then, p' is contained in $G \cap P$, and we may write $Y = G \cup P$. Moreover, $G \cap P$ is an annulus which is homotopy equivalent to S^1 . It follows that

$$\pi_1(Y, p') = \pi_1(G, p') *_{\pi_1(S^1, p')} \pi_1(P, p').$$

Moreover, $\tilde{\varphi}$ is homotopic to $\overline{p'p^{-1}\varphi p'p}$, so

$$\pi_1(Y, p') \cong \pi_1(P, p') / \langle \overline{p'p^{-1}\varphi p'p} \rangle.$$

But, note that P deformation retracts via $\mu_p \overline{p'p^{-1}}$ to give the desired result. $\square$

Question 6.10. What happens if we attach a k -cell for $k > 2$?

Claim 6.11. If $Y = X \sqcup D^k / \varphi(x) \sim x$ where $x \in \partial D^k$ for $k > 2$, then $\pi_1(Y) = \pi_1(X)$. That is, “attaching k -cells with $k > 2$ does not change the fundamental group”

Proof. Doing the same thing we did last time, by van Kampen's we have that

$$\pi_1(Y) = \pi_1(X) *_{\pi_1(S^{k-1})} \pi_1(D^k) = \pi_1(X)$$

as both $\pi_1(S^{k-1})$ and $\pi_1(D^k)$ are trivial. $\square$

Let X be a finite path connected CW-complex.

X^0 is just points. X^1 is a connected graph which is homotopic to $\bigvee_{i=1}^j S_i^1$ for some j , hence $\pi_1(X^1) = \mathbb{Z}^{*j}$.

X^2 is then obtained by attaching 2-cells via

$$X^2 = X_1 \coprod \coprod_{k=1}^{j_\alpha} \bar{D}_k^2 \left(\text{mod } x \sim \varphi_j^1(x) \right).$$

Choosing $x_0 \in X^1$, points $p_1, \dots, p_{j_\alpha} \in \partial D_k^2$ and curves μ_j connecting x_0 to p_j we have that

$$\pi_1(X^2, x_0) = \pi_1(X^1, x_0) / \langle\langle \mu_i^{-1} \varphi_j^1 \mu_i \rangle\rangle.$$

Now $\pi_1(X) = \pi_1(X^2)$.

Remark 6.12. *This is true without assuming finiteness.*

Exercise 10 (Important!). *Use van Kampen's theorem to compute*

- $\pi_1(\mathbb{T}^2)$,
- $\pi_1(\text{Klein Bottle})$,
- $\pi_1(\mathbb{RP}^2)$,
- $\pi_1(\mathbb{RP}^k)$ for all k .

7. Covering Spaces

Definition

7.1

Let X be path connected. A space $\tilde{X}$ with a continuous map $p : \tilde{X} \rightarrow X$ is called a **covering space** for X if for all $x \in X$ there exists an open set $U \ni x$ such that $p^{-1}(U) = \coprod_{\alpha \in I_x} U_\alpha$ for some index set I_x , where $p : U_\alpha \rightarrow U$ is a homeomorphism for all $\alpha \in I_x$.

Example 7.2. $X = S^1$, $\tilde{X} = \mathbb{R}$, $p(\theta) = e^{i\theta}$.

Example 7.3. $X = S^1$, $\tilde{X} = S^1$, by $z \mapsto z^k$ or equivalently $\theta \rightarrow k\theta$.

Example 7.4. $\mathbb{R}^2 \rightarrow \mathbb{R}^2 / \mathbb{Z}^2 \cong \mathbb{T}^2$ given by the natural projection.

Example 7.5. $\mathbb{R}^2 \rightarrow$ the Klein bottle given by the natural projection. "Think about it."

Example 7.6. We get an open cover of $S^1 \vee S^2$ by turning S^1 into a interval. Then we find another covering by "going around A at twice the speed." We of course want another covering, so we take the first covering and turn the remaining circles into intervals as well. This last covering is connected.

Homotopy Lifting Property

7.7

Suppose $\tilde{X} \rightarrow X$ is a covering map and Y is a space, $f_t : Y \times [0, 1] \rightarrow X$ is a homotopy (which

inputs like $f_5(y)$, and $\tilde{f}_0 : Y \rightarrow \tilde{X}$ is such that the following diagram commutes:

$$\begin{array}{ccc} & & \tilde{X} \\ & \nearrow \tilde{f}_0 & \downarrow p \\ Y & \xrightarrow{f_0} & X \end{array}$$

Then, there exists a unique homotopy $\tilde{f}_t$ such that

$$\begin{array}{ccc} & & \tilde{X} \\ & \nearrow \tilde{f}_t & \downarrow p \\ Y & \xrightarrow{f_t} & X \end{array}$$

commutes for all t .

Proof. Fix a point $y \in Y$, then $\text{Im}(\overbrace{t \mapsto f_t(y)}^{\gamma_y(t)})$ is compact and for each t we can choose $U_{f_t(y)} \subset X$ as in the definition of the covering space. (a neighborhood whose preimage under p is a disjoint union of itself). Now, $\{\gamma_y^{-1}(U_{f_t(y)})\}_{t \in [0,1]}$ is a open cover of $[0,1]$, which is compact so it has a Lebesgue number. Then, we want to construct $\tilde{\gamma}_y(t) := \tilde{f}_t(y)$, and to do this we can trace along the interval and watch where the path “must lift”.

Exercise 11. Prove $\tilde{f}_t$ is continuous.

□

Proposition

7.8

Let $x_0 \in X$ be any point. Then, $p^{-1}(x_0)$ is discrete, so in particular any path lifting the constant path $\gamma \equiv x_0$ is constant.

Proof. Trivial from the definition, but also follows by the unique homotopy lifting property (Proposition 7.7). □

Proposition

7.9

Suppose

$$(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$$

is a covering, with X path connected, and fix $[\gamma] \in \pi_1(X, x_0)$. Then, $[\gamma] \in p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$ if and only if γ lifts to a loop at $\tilde{x}_0$.

Proof. If γ lifts to a loop $\tilde{\gamma}$ in $\pi_1(\tilde{X}, \tilde{x}_0)$, then $\gamma = p \circ \tilde{\gamma}$, hence $p_*([\tilde{\gamma}]) = \gamma$. On the other hand, if $[\gamma] \in p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$, then γ is homotopic to $p \circ \tilde{\mu}$ for some loop $\tilde{\mu}$ at $\tilde{x}_0$ in $\tilde{X}$. Denoting by f_t this homotopy such that $f_0 = p \circ \tilde{\mu}$ and $f_1 = \gamma$, by the homotopy lifting property, there exists a unique $\tilde{f}_t$ lifting f_t such that $\tilde{f}_0 = \tilde{\mu}$.

$$\begin{array}{ccc}
 & \nearrow \tilde{\mu} & \tilde{X} \\
 [0, 1] & \xrightarrow{f_0 = p \circ \tilde{\mu}} & X \\
 & \downarrow p & \\
 & & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \nearrow \tilde{f}_t & \tilde{X} \\
 [0, 1] & \xrightarrow{f_t} & X \\
 & \downarrow p & \\
 & & X
 \end{array}$$

We claim $\tilde{f}_1$ is a loop at $\tilde{x}_0$ lifting γ . Indeed, as f_t is base-point preserving, $t \mapsto f_t(1)$ is a constant map. Hence, by [Proposition 7.8](#), f_t lifts to a constant path with

$$\tilde{f}_1(1) = \tilde{f}_0(1) = \tilde{\mu}(1) = \tilde{x}_0.$$

□

Question 7.10. When do two elements $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$ satisfy $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$ in $p^{-1}(x_0)$, assuming $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) = \tilde{x}_0$?

Well, knowing $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$, we may apply [Proposition 7.9](#), hence $\tilde{\gamma}_2^{-1} * \tilde{\gamma}_1$ is a loop at $\tilde{x}_0$ if and only if

$$[\gamma_2^{-1} * \gamma_1] \in p_*(\pi_1(X, x_0)) \iff \gamma_1 \in [\gamma_2] p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right),$$

i.e. when we have equivalence in the quotient space.

Corollary

7.11

$p^{-1}(x_0)$ is in bijective correspondence with

$$\pi_1(X, x_0) / p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right).$$

Question 7.12. When can f be lifted to a map

$$\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)?$$

That is, when can we make the following diagram commute?:

$$\begin{array}{ccc}
 & \nearrow \tilde{f} & (\tilde{X}, \tilde{x}_0) \\
 (Y, y_0) & \xrightarrow{f} & (X, x_0) \\
 & \downarrow p & \\
 & & (X, x_0)
 \end{array}$$

Theorem

7.13

Suppose Y is path connected and locally path connected. Then, a map $f : (Y, y_0) \rightarrow (X, x_0)$ can be lifted to $\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ if and only if

$$f_*(\pi_1(Y, y_0)) \subset p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$$

Proof. If a lift exists, i.e. there exists $\tilde{f}$ such that $p \circ \tilde{f} = f$, then

$$f_* = p_* \circ \tilde{f}_* \implies \text{Img}(f_*) \subset \text{Img}(p_*),$$

proving the first direction. For the other direction, fix $y \in Y$, and take a path γ connecting y_0 to y . Then, $f \circ \gamma$ is a curve in X starting at x_0 , so by the homotopy lifting property there exists $\tilde{\mu}(t)$ lifting $f(\gamma(t))$. Define $\tilde{f}_\gamma(y) := \tilde{\mu}(1)$. We now need to show that this $\tilde{f}$ is independent of the choice of γ .

Let γ_1, γ_2 be two such paths from y_0 to y . Define $\mu_i(t) = f(\gamma_i(t))$, and let $\tilde{\mu}_i$ be the lifts. We want to show $\tilde{\mu}_1(1) = \tilde{\mu}_2(1)$. Note that $\mu_1^{-1} \circ \mu_2$ is a loop in X at x_0 , and

$$[\mu_1^{-1} * \mu_2] = f_*([\gamma_1^{-1} * \gamma_2]) \in f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Thus, μ_1, μ_2 , and $\mu_1^{-1} * \mu_2$ have all been lifted to $\tilde{X}$, and these lifts are unique. It follows that $\tilde{\mu}_1(1) = \tilde{\mu}_2(1)$. Since we have well-definedness, we may set $\tilde{f}(y) = \tilde{f}_\gamma(y)$ for any γ .

Continuity is left to the reader. $\square$

Unique Lifting Property

7.14

Suppose Y is path connected, $f : Y \rightarrow X$ is continuous, and $\tilde{f}, \tilde{f}' : Y \rightarrow \tilde{X}$ are two lifts of f . If $\tilde{f}(y) = \tilde{f}'(y)$ for some $y \in Y$, then $\tilde{f} \equiv \tilde{f}'$.

Proof. Consider the set $A := \{y \mid \tilde{f}(y) = \tilde{f}'(y)\}$, which we assume is non-empty. It suffices to show that A is clopen. Fix $y \in A$, and select $U \ni y$ as in the definition of a covering space. Let $U_\alpha \subset \tilde{X}$ map homeomorphically to U via p , and such that $\tilde{f}(y) = \tilde{f}'(y) \in U_\alpha$. We claim $y \in f^{-1}(U) \subset A$. Indeed, since p is a homeomorphism from U_α to U , one sees that this is the case. $\square$

7.1. Covering Space / Subgroup Correspondence

Proposition

7.15

Let (X, x_0) be locally path connected, and suppose we have two coverings:

$$\begin{array}{ccc} (\tilde{X}, \tilde{x}_0) & & (\tilde{X}', \tilde{x}_0') \\ & \searrow p & \swarrow p' \\ & (X, x_0) & \end{array}$$

Moreover, suppose

$$p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right) \cong p'_*\left(\pi_1(\tilde{X}', \tilde{x}_0')\right).$$

Then, there exists a unique homeomorphism $f : (\tilde{X}, \tilde{x}_0) \rightarrow (\tilde{X}', \tilde{x}_0')$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & \tilde{X}' \\ & \searrow p & \swarrow p' \\ & X & \end{array}$$

Proof. Use [Theorem 7.14](#) to lift $p : \tilde{X} \rightarrow X$ to the cover $\tilde{X}'$, and to lift $p' : \tilde{X}' \rightarrow X$. These lifts are unique, and one can show they are inverses, hence the lift of p is the unique homeomorphism. $\square$

8. Universal Covers

Definition

8.1

A **universal cover** of X is a simply connected cover of X .

Question 8.2. Does there always exist a universal cover?

Definition

8.3

A space X is called **semi-locally simply connected** if for every point $x \in X$, there exists a neighborhood $U \ni x$ such that the image of the map $i_* : \pi_1(U, x) \rightarrow \pi_1(X, x)$ induced by the embedding $i : U \hookrightarrow X$ is trivial.

Remark 8.4. Simply connected implies semi-locally simply connected.

Proposition

8.5

If X has a universal cover, then X is semi-locally simply connected.

Proof. Fix $x \in X$, and let $U \ni x$ be as in the definition of the covering space $\tilde{X}$, which is our universal covering. Let $\tilde{x} \in \tilde{X}$ be such that $p(\tilde{x}) = x$, and let $U_\alpha \ni \tilde{x}$ be such that $p : \tilde{U}_\alpha \rightarrow U$ is a homeomorphism. Then, any loop γ at x in U lifts to a loop $\tilde{\gamma}$ at $\tilde{x}$ in $\tilde{U}_\alpha$. However, since $\tilde{U}_\alpha$ is simply connected by our assumption, we know $\tilde{\gamma}$ is homotopically trivial, and thus γ is as well. $\square$

Existence of the Universal Cover

8.6

Suppose X is path connect, locally path connected, and semi-locally simply connected. Then, X has a universal cover.

Proof. We want to construct the universal cover $\tilde{X}$. Suppose, for now, that such a universal cover existed, and $p : \tilde{X} \rightarrow X$ is the projection. Fix $x_0 \in X$, and $\tilde{x}_0 \in \tilde{X}$. Given a curve γ at x_0 , we can lift it to $\tilde{\gamma}$ starting at $\tilde{x}_0$. Now, consider

$$\Gamma := \{\gamma : [0, 1] \rightarrow X, \gamma(0) = x_0\}.$$

Then, we have an induced map $F : \Gamma \rightarrow \tilde{X}$ given by $F(\gamma) := \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the correspond lift. Since $\tilde{X}$ is path connected, F is surjective. Moreover, if $F(\gamma_1) = F(\gamma_2)$, we must have $\gamma_1(1) = \gamma_2(1)$, and $\tilde{\gamma}_2^{-1} \circ \tilde{\gamma}_1$ is a loop in $\tilde{X}$ and therefore homotopically trivial, which means $\gamma_2^{-1} \circ \gamma_1$ are as well. That is, $\gamma_1 \stackrel{\text{rel}\{0,1\}}{\sim} \gamma_2$. Conversely, if $\gamma_1 \sim \gamma_2$, then by the homotopy lifting property, $F(\gamma_1) = F(\gamma_2)$. Thus, F induces a bijection $(\Gamma / \sim) \leftrightarrow \tilde{X}$, where $\sim$ is homotopy relative to $\{0, 1\}$. This hints that, to find our universal cover $\tilde{X}$, we should use the collection $\Gamma / \sim$.

Indeed, define

$$\tilde{X} := \{[\gamma]_{\text{rel}\{0,1\}} \mid \gamma : [0, 1] \rightarrow X, \gamma(0) = x_0\},$$

and let $p : \tilde{X} \rightarrow X$ be the projection $p([\gamma]) = \gamma(1)$, which is well-defined. Define a basis B for X

by

$$B := \{U \text{ path connected and } i_*(\pi_1(U, x_0)) \subset \pi_1(X, x_0) \text{ is trivial}\}.$$

Note that if $V \subset U \in B$ is path-connected, then $V \in B$. Thus, if $U_0 \cap U_1$ is non-trivial for $U_0, U_1 \in B$, then for any $x \in U_0 \cap U_1$, there exists a path connected $V \ni x_0$, $V \subset U_0 \cap U_1$, because X is locally path connected, and then $V \in B$. Moreover, B covers X , hence B indeed defines a basis for X .

Now, to obtain a topology for $\tilde{X}$: for every $[\gamma] \in \tilde{X}$ and $U \in B$ such that $\gamma(1) \in U$, take

$$U_{[\gamma]} := \{[\mu \circ \gamma] \mid \mu(0) = \gamma(1), \text{Im}(\mu) \subset U\}.$$

Then, p induces a bijection $U_{[\gamma]} \rightarrow U$ for similar reasons as before. Moreover, if $\gamma_2 \in U_{[\gamma]}$, then $U_{[\gamma_2]} = U_{[\gamma]}$. Thus

$$\tilde{B} := \{U_{[\gamma]} \mid U \in B, \gamma(1) \in U\}$$

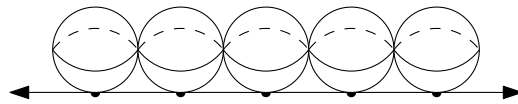
defines a basis for $\tilde{X}$: if $U_{[\gamma_1]}^1 \cap U_{[\gamma_2]}^2 \neq \emptyset$, then we may take γ_3 in the intersection so that $U_{[\gamma_1]}^1 = U_{[\gamma_3]}^1$ and $U_{[\gamma_2]}^2 = U_{[\gamma_3]}^2$. Thus, since B is a basis for U , we may take V with $\gamma_3(1) \in V \subset U_1 \cap U_2$, and then $V_{[\gamma_3]} \subset U_{[\gamma_3]}^1 \cap U_{[\gamma_3]}^2 \in \tilde{B}$, proving $\tilde{B}$ is a basis.

One now checks that $\tilde{X}$, with the topology induced by $\tilde{B}$, indeed defines a covering. Finally, to prove that it's simply connected: fix $[\gamma]$ a curve on X , and define

$$[\gamma]_t(s) := \begin{cases} \gamma(s) & 0 \leq s \leq t \\ \gamma(t) & t \leq s \leq 1 \end{cases},$$

which demonstrates that $\tilde{X}$ is path-connected. Moreover, if $\gamma \in p_*(\pi_1(X, x_0))$, and $\tilde{\gamma}$ is the lift, then we know that $[\gamma] = [x_0]$, and so $\tilde{\gamma}$ is homotopically trivial, proving simply connectedness. $\square$

Example 8.7. The universal cover of S^1 adjoined to S^2 is $\mathbb{R}$ with a copy of S^2 at every $2\pi h$, $h \in \mathbb{Z}$.



Exercise 12. Compute the universal cover of $S^1 \vee T^2$.

Proposition

8.8

If $\tilde{X}$ is a cover of X , and $x_0 \in X$, then $\pi_1(X, x_0)$ acts on $p^{-1}(x_0)$: given $\tilde{x}_0 \in p^{-1}(x_0)$, and $\gamma \in \pi_1(X, x_0)$, we may lift γ to a curve $\tilde{\gamma}$ in $\tilde{X}$ starting at $\tilde{x}_0$, and so we define the action of γ on $\tilde{x}_0$ to be $\tilde{\gamma}(1)$.

Lemma

8.9

If $\tilde{X}$ is the universal cover of X , then $\pi_1(X, x_0)$ acts on $\tilde{X}$, extending the above action in a covering respecting manner: for all $g \in \pi_1(X, x_0)$, and $\tilde{x} \in \tilde{X}$, $p(g \cdot \tilde{x}) = p(\tilde{x})$.

Proof. Take $\tilde{x}_0 \in p^{-1}(x_0)$. Given $g \in \pi_1(X, x_0)$. We have the induced projections

$$\begin{array}{ccc} & (\tilde{X}, g\tilde{x}_0) & \\ & \downarrow p & \\ (\tilde{X}, \tilde{x}_0) & \xrightarrow{p} & (X, x_0) \end{array}$$

and so we want to find a continuous map T_g between $(\tilde{X}, \tilde{x}_0)$ and $(\tilde{X}, g\tilde{x}_0)$.

Given $\tilde{x} \in \tilde{X}$, take a curve $\tilde{\mu}$ from $\tilde{x}_0$ to $\tilde{x}$, and so consider $\mu := p \circ \tilde{\mu}$, which we lift back up to $\tilde{X}$ starting at $g(\tilde{x}_0)$. It's left as an exercise to show that this is a group action. $\square$

Proposition

8.10

Let $H \leq \pi_1(X, x_0)$ be a subgroup, and denote $X_H := \tilde{X}/H$, the space modded out by induced action of H . Then X_H is a covering space of X , with projection $p_H : X_H \rightarrow X$, and

$$p_{H*}(\pi_1(X_H, [\tilde{x}_0])) = H$$

Proof. We have $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$, the quotient map $\tilde{p} : (\tilde{X}, \tilde{x}_0) \rightarrow (X_H, x_0^H)$, where $x_0^H := [\tilde{x}_0]$, and we have $p_H : (X_H, x_0^H) \rightarrow (X, x_0)$. We claim p_H is a covering map.

Let $x \in X$, and let $U \ni x$ be path connected such that $p^{-1}(U) = \coprod U_\alpha$, with $p : U_\alpha \rightarrow U$ a homeomorphism.

Claim 8.11. For $g \in \pi_1(X, x_0)$, if $g(U_\alpha) \cap U_\beta \neq \emptyset$, then $g(U_\alpha) = U_\beta$.

Indeed, let $\tilde{z}_0 \in U_\alpha$ be such that $g\tilde{z}_0 \in U_\beta$. Since U_α and U are path connected, we may take $z_0 := p(\tilde{z}_0)$, and then for all $\tilde{z} \in U_\alpha$, we take a path $\tilde{\gamma}$ in U_α connection $\tilde{z}_0$ and $\tilde{z}$. Take $\gamma := p \circ \tilde{\gamma}$, and lift γ to a path starting at $g\tilde{z}_0$. We claim that $g \cdot \tilde{\gamma}$ is such a lift. Thus, $g\tilde{z} \in U_\beta$. Show the reverse inclusion finishes the claim.

Continued next time. $\square$

Testing references to non-boxes:

[Exercise 12](#)

[Example 8.7](#)